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LARGE AMPLITUDE WAVES IN BOUNDED MEDIA

II.† THE DEFORMATION OF AN IMPULSIVELY LOADED SLAB: THE FIRST REFLEXION

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CONTENTS

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This paper continues the investigation of large amplitude waves in bounded media started by Cekirge & Varley (1973). It describes the early stages of the deformation produced in an elastic slab contained between two parallel, plane interfaces when the normal traction at one of them changes discontinuously. During the subsequent deformation energy is radiated across these interfaces to adjacent elastic materials. Typically, the disturbance in the slab could be caused by the arrival of a constant strength shock wave travelling through an adjacent material or when the slab, which forms the front part of a composite material, impacts some other elastic material.

It is assumed that the dynamic response of the slab can be approximated by that of one of the model materials introduced in the first part of this study. It is shown that this is possible for a whole host of materials. These include polycrystalline solids, metals when subjected to high pressure, water, explosive products, gases, yarns as well as elastic–plastic, rigid–plastic and rigid–elastic materials. The results reported are obtained by showing that for these model materials a simple, but exact, representation can be found that describes the interaction of a centred wave with *any* wave travelling in the opposite direction. The arbitrary functions occurring in this representation are then found for the special case when this opposite travelling wave is the wave reflected from an interface with some other elastic material during the arrival of the centred wave. The limiting cases of a perfectly free interface, a perfectly rigid interface, and an interface with a Hookean material are analysed in great detail.

Although the terminology used in this paper is that associated with nonlinear elastodynamics, the results are directly applicable to any system whose response is described by the nonlinear wave equation. For example, the slab could represent a layer of nonlinear dielectric embedded in some other nonlinear dielectric and the disturbance could be generated by the arrival of an electromagnetic shock. Alternatively, the slab could represent sea water which is bounded by air from above and by rock from below while the disturbance is produced by a sudden motion of the water/rock interface.

1. INTRODUCTION

This paper is part II of a study of large amplitude waves in bounded media. Part I, an earlier paper by Cekirge & Varley (1973), studied the deformation that is produced when a large amplitude, but shockless, pulse travelling through an elastic material arrives at an interface with another elastic material. The pulse propagates in a direction normal to the interface. First, the ideas of nonlinear impedance, reflexion coefficient and transmission coefficient for such an interface were introduced. These were then used to determine the amplitudes of the reflected and transmitted pulses in terms of that of the incident pulse. The algorithms obtained were quite general: no restriction was placed on the stress–strain relations of the materials separated by the interface. As an illustration of the use of these algorithms, the decay in the amplitude of a pulse was calculated as it bounces back and forth in a slab, or layer, contained between two other elastic materials. At each contact with an interface part of the energy of the pulse is transmitted into the surrounding material and part is reflected. Of course, the information that can be obtained by using these general results is rather limited (although, in practice, it may be sufficient). In fact, just a detailed calculation of the deformation produced at an interface during the arrival of a single pulse is difficult since it involves the nonlinear interaction of the incident and reflected pulses. Usually, this problem can only be tackled by using numerical procedures. However, in part I it was shown that when the material through which the incident wave travels belongs to a certain family of model materials the deformation at the interface during the arrival of *any* pulse can easily be analysed. The elastic material which is separated by the interface from the model material may be quite arbitrary.

The model materials are defined by the fact that A, the Lagrangian sound speed, varies with the strain e according to the law

$$\mathrm{d}A/\mathrm{d}e = \mu A^{\frac{3}{2}}(1-A/M), \tag{1.1}$$

where μ and M are material constants. Some of the different kinds of material responses that can be modelled by such relations are catalogued in part I. They range from elastic–plastic to elastic–locking materials. Here, in § 2 of this paper, we extend this list and show how to choose the parameters μ and M, together with the constants of integration for equation (1.1), so that the corresponding stress–strain relations best approximate the stress–strain relations of a wide variety of real materials (see tables 1–3). These include polycrystalline solids, yarns and silks, metals when subjected to pressures of up to *ca.* 10^{10} Pa (10^5 atm), explosive products, water, and many gases. With the exception of gases, the curve fit provided by the model materials is as good a fit to the experimental data as any we can find in the literature. For gases the relative error in the approximation to $A(e)$ is less than 2 % as the density changes by a factor of ten.

In all of the deformations studied in this paper the reference state R, from which the strain e and the traction T are measured, is that in which A is a maximum, $= A_0$ say. Since $A^2 \propto \mathrm{d}T/\mathrm{d}e$, this means that in all deformations the material softens relative to its state R. (For gases, R is the state of maximum pressure and density.) This reference state need not, however, be an equilibrium state of the material. Sometimes it is the state induced by the passage of a constant strength shock.

There are two distinct classes of model materials that soften relative to R. We call these non-ideal and ideal soft materials. If A is measured in units of A_0, so that $A \leqslant 1$ for the deformations studied here, M lies in the range $[0, 1]$ for non-ideal materials and outside this range for ideal materials. For the former $\mu e \geqslant 0$ and, as $\mu e \to \infty$, $A \to M$: for the latter $\mu e \leqslant 0$ and, as $\mu e \to -\infty$, $A \to 0$. In § 2 it is shown that the behaviour of many polycrystalline materials during uniaxial compression can be modelled by those of non-ideal materials. On the other hand, the behaviour of gases, explosive products, water and metals when subjected to high pressure can be modelled by ideal materials. For gases M lies in the range $(-\infty, 0)$; for metals, water, and explosive products M lies in the range $(1, \infty)$. For all these materials the approximations are only valid when μe varies over a *finite* range.

In addition to the materials listed in tables 1–3, four other highly idealized responses which can be obtained as limiting cases of equation (1.1) are also discussed in § 2. Two of these were pointed out in part I. These were the simple linear Hookean response, which is obtained as $M \to 1$ from values < 1 while μ is held constant, and the linear elastic–perfectly plastic response, which is obtained as $M \to 1$ from values > 1 while $\mu/\ln(M-1)$ is held constant. For example, in the latter limit the dominant variation of A with T is expressed by the formula

$$A = \tanh^2[\eta_0(1 - T/T_1)] \quad \text{as} \quad \eta_0 = -\tfrac{1}{2}\ln(M-1) \to \infty, \tag{1.2}$$

where T_1 is the limiting stress of the material. The other two limiting responses introduced in this paper are obtained as $M \to 0$. As $M \to 0$ from values > 0 while μ/M is held constant the perfectly rigid–perfectly elastic response is obtained (see figure 3). As $M \to 0$ from values < 0 while μ is held constant the perfectly rigid–perfectly plastic response is obtained (see figure 3). These limiting responses can be used to approximate the behaviour of yarns and strings in tension as well as the behaviour of concrete in compression.

One of the limitations of the model materials discussed here is that they can only be used to approximate the response of materials over ranges where A is either a monotonically increasing or a monotonically decreasing function of e. They cannot be used to approximate any stress–strain relation that contains a point of inflexion. Such a stress–strain relation can occur, for example, when a cylinder of polycrystalline material is compressed by an axial load. Initially the load is mainly balanced by the material's resistance to shear: over this range the material softens for the applied load. However, when the load is sufficiently large it is mainly balanced by the material's resistance to compression: over this range the material hardens. Consequently, any stress–strain relation that is uniformly valid in compression contains a point of inflexion. Recently, we have been able to generalize the class of model materials for which the governing equations can be integrated so that the effect of a point of inflexion can be analysed for some deformations of practical importance. However, the representations of these deformations are so complex compared with those that are obtained when equation (1.1) holds that we do not discuss them here.

After establishing the fact that the model materials are of some physical relevance, in the remainder of the paper we analyse the deformation produced in a slab of such a material when a centred wave travelling through the slab is reflected from an interface with some other elastic material. Various situations in which this problem occurs are described in §3. One example is the initial stage of the deformation produced in a slab when a constant strength shock wave travelling through an adjacent material arrives at their common interface. During the subsequent deformation energy is radiated across the other boundary of the slab to some other adjacent material. The slab could be a panel surrounded on both sides by air, or it could be an element of some composite material. Alternatively, the slab could be sea water which is bounded by air from above and by rock from below. The disturbance in the water is produced by the arrival of a constant strength shock wave travelling through the rock. Another example of a deformation that involves the reflexion of a centred wave from an interface occurs when a 'hitter' bar, or plate, which is a composite of two materials impacts some other elastic material. A much simpler, but more idealized, example is the deformation produced in a bar, or string, when one of its ends is suddenly loaded by a traction, or tension, which is then held constant for some time while the other end is held fixed.

The main result of this paper is obtained in §4. There, a simple representation (see equations (4.8) and (4.9)) is obtained that describes the deformation produced when a centred wave interacts with *any* other wave travelling in the opposite direction. In the remainder of the paper we

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calculate the arbitrary functions occurring in this representation, and analyse the corresponding deformations, in the special case when the opposite travelling wave is that reflected from an interface with some other elastic material during the arrival of the centred wave. Before the arrival of the incident centred wave the material is in the uniform reference state R. During its passage the material softens.

As a first illustration of the use of the representation obtained in §4, in §2 we consider the limiting case when the interface is perfectly free. The main effect of the wave that is reflected from such an interface is to harden, or unload, the material. This causes one of two things to happen: the material either stops behaving elastically or the reflected wave focuses to form a shock. The results obtained in §6 are only valid for materials that continue to behave elastically on unloading and then only up until the time when a shock forms. (Inelastic behaviour and the effect of shocks will be considered in subsequent papers.) With these limitations in mind, in the interaction region a simple representation is obtained for A as an explict function of the Lagrangian distance measure X, time measure t and the material parameter M (see equation (6.9)). The behaviour of a whole range of different materials is obtained by simply letting M take on different values. The only time the other material parameters enter is when T, e and the material velocity u are calculated. The details of the deformation are calculated and summarized by graphs in the limiting cases when the material response is Hookean, elastic–plastic, rigid–plastic, and rigid–elastic.

The reflexion from a rigid interface is described in §7. Now the effect of the reflected wave is to further soften the material so that no shocks form and no unloading occurs. In the interaction region A can be expressed as the ratio of two quadratic forms in X and t with coefficients that depend on M. The corresponding deformations are analysed in detail. In particular, the strength of the incident wave that causes the material to yield at the rigid interface is calculated.

In many situations the interface from which the centred wave is reflected separates the slab from some other elastic material whose response is essentially linear for the stress level that occurs at the interface, although that of the slab is grossly nonlinear. In §8 we describe the deformation produced in the slab during the reflexion of a centred wave from such an interface. Now, in addition to the material parameter M, A depends on i_0, the impedance of the interface when the material is in the reference state R. Except when $i_0 = 0$, which corresponds to a perfectly free interface, the material always softens at the interface. This causes the reflected wave to defocus. However, when the material is ideally soft and when $i_0 \leqslant 1$ the material may begin to harden away from the interface and, if the material continues to behave elastically, the reflected wave may focus to form a shock. Hardening and shock formation always occur first at the front of the reflected wave. Several important features of the deformation are analysed, including the strength of the incident centred wave, which now depends on i_0 as well as M, that causes the material to yield.

Finally, in §9, we show how to calculate the functions that occur in the representation obtained in §4 when the interface separates the slab from any other elastic material. After crossing the interaction region the reflected wave emerges as a simple wave. The procedure that must be used to calculate the deformation during its passage is described.

2. THE GOVERNING EQUATIONS

In this section we list those relations which were established in part I of this paper that will be needed here.

Characteristic variables (α, β) are used as independent variables. Then, the equations governing uni-axial isentropic, stretching waves in an elastic material imply that the Lagrangian distance measure $X(\alpha, \beta)$ and the time measure $t(\alpha, \beta)$ satisfy the equations

$$\partial X/\partial\beta = A\,\partial t/\partial\beta \quad \text{and} \quad \partial X/\partial\alpha = -A\,\partial t/\partial\alpha. \tag{2.1}$$

In these equations, A, the Lagrangian sound speed, is regarded as a function, $\hat{A}(c)$, of a variable c that can be expressed in terms of the signal functions $F(\alpha)$ and $G(\beta)$ as

$$c = F(\alpha) + G(\beta). \tag{2.2}$$

The function $\hat{A}(c)$ determines the dynamic response of the material: in terms of it

$$\text{the traction} \quad T = \rho_0 \int_0^c \hat{A}(s)\,\mathrm{d}s \quad \text{and the strain} \quad e = \int_0^c \frac{\mathrm{d}s}{\hat{A}(s)}, \tag{2.3}$$

where ρ_0 is the constant density of the material when $c = 0$. In terms of F and G,

$$\text{the material velocity} \quad u = G(\beta) - F(\alpha). \tag{2.4}$$

For the materials studied here, A and c are related by the equation

$$\mathrm{d}A/\mathrm{d}c = \mu A^{\frac{1}{2}} + \nu A^{\frac{3}{2}}, \tag{2.5}$$

where μ and ν are material constants. There are two reasons for choosing equations of state for which $\hat{A}(c)$ satisfies equation (2.5). First, the hodograph equation for t, which is obtained by eliminating X from equations (2.1), can be integrated. Second, these model stress–strain relations can be used to approximate the actual stress–strain relations of a wide variety of real materials with quite remarkable accuracy.

2.1. *Integration of the hodograph equations*

Throughout this paper the elastic material will be bounded by the material planes $X = 0$ and $X = D$. Then, as was shown in part I, it follows from equations (2.1), (2.2) and (2.5) that

$$2A^{\frac{1}{2}}\,\partial t/\partial\alpha + [\mu(t-\alpha) + \nu X]\,F'(\alpha) = m(\alpha), \tag{2.6}$$

and that

$$2A^{\frac{1}{2}}\,\partial t/\partial\beta + [\mu(t-\beta) + \nu(D-X)]\,G'(\beta) = n(\beta). \tag{2.7}$$

In these equations, the characteristic variables (α, β) have been normalized so that

$$\text{when} \quad X = 0, \quad \alpha = t, \tag{2.8}$$

while

$$\text{when} \quad X = D, \quad \beta = t. \tag{2.9}$$

Also,

$$m(t) = A^{\frac{1}{2}} \quad \text{at} \quad X = 0 \quad \text{while} \quad n(t) = A^{\frac{1}{2}} \quad \text{at} \quad X = D. \tag{2.10}$$

This indentification follows from equations (2.6)–(2.9) and the facts [I, (7.13)] that at any particle X

$$\mathrm{D}t/\mathrm{D}\alpha = 2\,\partial t/\partial\alpha \quad \text{and} \quad \mathrm{D}t/\mathrm{D}\beta = 2\,\partial t/\partial\beta. \tag{2.11}$$

Once the functions F, G, m and n have been calculated from prescribed initial and boundary data, the relations (2.1), (2.6) and (2.7) can be used to determine $X(\alpha, \beta)$ and $t(\alpha, \beta)$. Before the general procedure for doing this is described though, we first show how these relations can be used to calculate some comparatively simple deformations that involve wave interactions.

2.2. *Stress–strain relations for ideally soft materials*

For most of the deformations studied in this paper the material is in a uniform state, R, at $t = 0$, where $c = e = T = 0$ and $A = A_0$. Subsequently,

$$\text{for} \quad t \geqslant 0, \quad A \leqslant A_0, \tag{2.12}$$

so that the material *softens* relative to its state at $t = 0$. However, because in some of the deformations the state R is induced by the passage of a constant strength shock, u is not necessarily zero at $t = 0$.

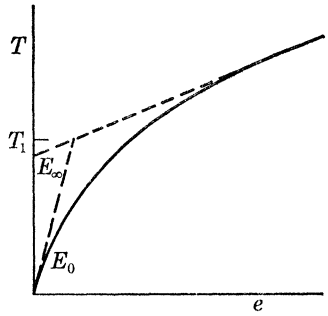

FIGURE 1

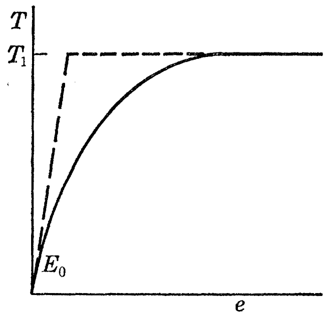

FIGURE 2

FIGURE 1. A typical stress–strain relation for an ideally soft elastic material. The sound speed A decreases monotonically with either increasing or decreasing e from $A_0 = \sqrt{(E_0/\rho_0)}$ at $e = 0$ to zero. The values of E_0, the limiting stress T_1, and the value of e at which $T = 0.99\,T_1$ can be specified.

FIGURE 2. A typical stress–strain relation for a soft material. The sound speed A decreases monotonically with either increasing or decreasing e from $A_0 = \sqrt{(E_0/\rho_0)}$ at $e = 0$ to some limiting value $A_\infty = \sqrt{(E_\infty/\rho_0)}$. The values of E_0, E_∞ and T_1 can be specified.

According to equations (2.3) and (2.5), materials that soften are of two distinct types: ideal or non-ideal. A typical stress–strain relation for an ideally soft elastic material is shown in figure 1. If, for example, the material softens as e increases, as it does for a gas, A decreases monotonically from A_0 to zero as e increases from zero to infinity. The material parameters (μ, ν) can be chosen so that both T_1, the limiting value of T as $A \to 0$, and ϵ, the value of e at which $T = 0.99T_1$, can be specified.

There are two families of solutions to equation (2.5) that describe ideally soft materials. For the first family

$$A/A_0 = M \tanh^2[\eta_0 + \eta_1(c/A_0)], \tag{2.13}$$

where $$\mu A_0^{\frac{1}{2}} = 2\eta_1 M^{\frac{1}{2}}, \quad \nu A_0^{\frac{3}{2}} = -2\eta_1 M^{-\frac{1}{2}} \quad \text{and} \quad 1 \leqslant M \leqslant \infty. \tag{2.14}$$

For the second family $$A/A_0 = -M \tan^2[\theta_0 + \theta_1(c/A_0)], \tag{2.15}$$

where $$\mu A_0^{\frac{1}{2}} = 2\theta_1 |M|^{\frac{1}{2}}, \quad \nu A_0^{\frac{3}{2}} = 2\theta_1 |M|^{-\frac{1}{2}} \quad \text{and} \quad -\infty \leqslant M \leqslant 0. \tag{2.16}$$

In practice, of course, the parameters in the expressions (2.13) and (2.15) can only be chosen so that these expressions approximate the actual variation of A with c in a real material over part of the range $(A_0, 0)$. For example, table 1 shows how to choose the parameters M, η_0 and η_1 in equation (2.13) to curve fit the responses of many metals in the hydrodynamic range as the pressure varies from a value which is large compared with the yield stress to a value of the order of 10^{10} Pa (10^5 atm). Over this range the experimental pressure–density relation ($p_E - \rho_E$ relation†) is often (Al'tshuler 1965) approximated by the law

$$p_E = p_0 + \frac{\rho_0 A_0^2}{\gamma}\left[\left(\frac{\rho_E}{\rho_0}\right)^\gamma - 1\right], \tag{2.17}$$

where the constant exponent γ lies somewhere in the range $4 \leqslant \gamma \leqslant 5$. In equation (2.17), without loss of generality, p_0 and ρ_0 denote the maximum pressure and density that occur in the deformation. Equation (2.17) is also used (Cole 1948) to describe the dynamic response of water over a pressure range of 10^9 Pa (10^4 atm): then $7 \leqslant \gamma \leqslant 8$. Also, equation (2.17), with $\gamma \simeq 3$, is often used (Baum 1959) to describe the responses of dense gases, such as explosive products. In terms of

$$T_E = p_0 - p_E \quad \text{and} \quad e_E = \rho_0/\rho_E - 1, \tag{2.18}$$

equation (2.17) can be written

$$T_E = \frac{\rho_0 A_0^2}{\gamma}[1-(1+e_E)^{-\gamma}]. \tag{2.19}$$

This corresponds to

$$A_E/A_0 = [1-\tfrac{1}{2}(\gamma-1)\,(c/A_0)]^{(\gamma+1)/(\gamma-1)}. \tag{2.20}$$

Table 1. The values of M, η_0 and η_1, and the corresponding values of (μ, ν), for which the theoretical law $A/A_0 = M\tanh^2[\eta_0+\eta_1(c/A_0)]$ best fits the experimental law $A_E/A_0 = [1-\frac{1}{2}(\gamma-1)\,(c/A_0)]^{(\gamma+1)/(\gamma-1)}$ over the density range $(\rho/\rho_0)_{\min} \leqslant \rho/\rho_0 \leqslant 1$. The relative error over this range is less than 2 %

	material				
	explosion products	metals		water	
parameters	$\gamma = 3$	$\gamma = 4$	$\gamma = 5$	$\gamma = 7$	$\gamma = 8$
M	$+\infty$	4.6316	3.0362	2.4855	2.1768
η_0	0	0.5004	0.6408	0.7423	0.8146
η_1	0	−0.7009	−1.1726	−1.9197	−2.4536
$\mu A_0^{\frac{1}{2}}$	−2	−3.0168	−4.0864	−6.0530	−7.2401
$\nu A_0^{\frac{3}{2}}$	0	0.6514	1.3459	2.4353	3.3260
$(\rho/\rho_0)_{\min}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$

Table 1 displays the values of M, η_0 and η_1, when $\gamma = 3, 4, 5, 7, 8$, for which $A(c)$, given by equation (2.13), approximates $A_E(c)$, given by equation (2.20), over a specified range of ρ/ρ_0. When $\gamma = 3$, $A(c) \equiv A_E(c)$ because

$$A/A_0 = (1-c/A_0)^2 \quad (\gamma = 3) \tag{2.21}$$

is an exact solution of equation (2.5) with

$$\mu = -2A_0^{-\frac{1}{2}} \quad \text{and} \quad \nu = 0. \tag{2.22}$$

The expression (2.21) is obtained from (2.13) in the limit as

$$(\eta_0, \eta_1) \to 0, \quad M \to \infty, \quad \eta_0 M^{\frac{1}{2}} \to 1 \quad \text{and} \quad \eta_1 M^{\frac{1}{2}} \to -1. \tag{2.23}$$

For $\gamma = (4, 5)$ the relative error in approximating A_E by A over the range $\frac{1}{2} \leqslant \rho/\rho_0 \leqslant 1$ is less than 2 %. For $\gamma = (7, 8)$, over the range $\frac{1}{3} \leqslant \rho/\rho_0 \leqslant 1$, it is also less than 2 %. Both these ranges

† The subscript E denotes the exact or the experimental value.

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cover the ranges over which the expression (2.20) itself provides an adequate approximation. Over these ranges, $A(c)$ is just as good an approximation as $A_E(c)$ to the experimental data.

Other material behaviours that can be modelled by the equations of state (2.13) are discussed in part I of this paper. Here, for future reference, we summarize what happens when

$$M \to 1, \quad \eta_0 \to \infty \quad \text{and} \quad \eta_1/\eta_0 \to -\rho_0 A_0^2/T_1. \tag{2.24}$$

In this limit, according to equations (2.13) and (2.24), the dominant behaviour is described by the equations

$$A/A_0 = \tanh^2[\eta_0(1-T/T_1)]/\tanh^2\eta_0, \tag{2.25}$$

where
$$\frac{\rho_0 A_0}{T_1}c = \frac{T}{T_1} \quad \text{and} \quad \frac{\rho_0 A_0^2}{T_1}e = \frac{T}{T_1} + \eta_0^{-1}\coth\left(\eta_0\left[1-\frac{T}{T_1}\right]\right). \tag{2.26}$$

As $\eta_0 \to \infty$, equations (2.25) and (2.26) predict that the prefectly elastic–perfectly plastic response is obtained:

$$A \to \begin{cases} A_0 & \text{for} \quad 0 \leqslant T/T_1 < 1, \\ 0 & \text{for} \quad T/T_1 = 1. \end{cases} \tag{2.27}$$

Table 2. The values of M, θ_0 and θ_1, and the corresponding values of (μ, ν), for which the theoretical law $A/A_0 = M\tan^2[\theta_0 + \theta_1 (c/A_0)]$ best fits the experimental law $A_E/A_0 = [1-\frac{1}{2}(\gamma-1)(c/A_0)]^{(\gamma+1)/(\gamma-1)}$ over the density range $(\rho/\rho_0)_{\min} \leqslant \rho/\rho_0 \leqslant 1$. The relative error over this range is less than 2 %

	gases				
parameters	$\gamma = 1$	$\gamma = 1.4$	$\gamma = \frac{5}{3}$	$\gamma = 2$	$\gamma = 3$
M	−0.4143	−0.6479	−0.8081	−0.5300	$-\infty$
θ_0	0.9989	0.8928	0.8420	0.6814	0
θ_1	−0.2409	−0.3248	−0.3813	−0.3985	0
$\mu A_0^{\frac{1}{2}}$	−0.3101	−0.5221	−0.6856	−0.9858	−2
$\nu A_0^{\frac{3}{2}}$	−0.7484	−0.8006	−0.8484	−0.6443	0
$(\rho/\rho_0)_{\min}$	0.1	0.1	0.1	0.1	0

It was shown in part I that the second family of ideally soft materials, for which A is given by equation (2.15), can be used to approximate the responses of gases as the density changes by a factor of ten. Over this range $A_E(c)$ is given by equation (2.20) with γ lying in the range $1 \leqslant \gamma \leqslant 3$. For completeness, table 2 displays the values of M, θ_0 and θ_1 when $\gamma = 1, \frac{7}{5}, \frac{5}{3}$, 2 and 3 for which $A(c)$, given by equation (2.15), best approximates $A_E(c)$. Again the maximum relative error is about 2 %. Note that the case $\gamma = 3$ is again obtained from the expression (2.15) as $M \to -\infty$.

The second family of ideally soft materials can also be used to model the behaviour of materials that are essentially rigid when $0 \leqslant T/T_1 \leqslant 1$ but which soften rapidly as $T \to T_1$. This behaviour is obtained as

$$\epsilon = \frac{2}{\pi}\cot\theta_0 = \frac{2}{\pi}(-M)^{\frac{1}{2}} \to 0, \quad \epsilon A_0 \to \frac{T_1}{\rho_0 c_1} \tag{2.28}$$

and
$$\epsilon\theta_1 \to -\frac{T_1}{\rho_0 c_1^2}\tfrac{1}{2}\pi,$$

where c_1 is an arbitrary reference velocity. In this limit, according to equations (2.15) and (2.3), the dominant behaviour is described by the equations

$$\frac{\rho_0 c_1}{T_1}A = \epsilon\left[\epsilon + \frac{2}{\pi}\tan\bar{c}\right]^{-2} \quad \text{and} \quad \frac{T}{T_1} = \tan\bar{c}[\tan\bar{c} + \tfrac{1}{2}\pi\,\epsilon]^{-1} \quad \text{where} \quad \bar{c} = \tfrac{1}{2}\pi\,(c/c_1). \tag{2.29}$$

Also, $$\frac{T_1}{\rho_0 c_1^2}e = \left(\frac{2}{\pi}\right)^3 \epsilon^{-1}\left[\frac{\tan\bar{c}}{1-\frac{1}{2}\pi\tan\bar{c}} - \bar{c}\right], \quad \text{where} \quad 0 \leqslant \bar{c} \leqslant \tfrac{1}{2}\pi(1-\epsilon). \tag{2.30}$$

The rigid response is described by equations (2.29) and (2.30) in a layer neighbouring $c = 0$, where $c/c_1 = O(\epsilon)$. In this layer, if we write

$$c/c_1 = \epsilon\eta, \tag{2.31}$$

equations (2.29) and (2.30) imply that

$$\frac{\rho_0 c_1}{T_1}A = \epsilon^{-1}(1+\eta)^{-2}, \quad \frac{T}{T_1} = \frac{\eta}{1+\eta} \tag{2.32}$$

and $$\frac{T_1}{\rho_0 c_1^2}e = \epsilon^2[\eta+\eta^2+\tfrac{1}{3}\eta^3]. \tag{2.33}$$

According to equations (2.31)–(2.33) when $c/c_1 = O(\epsilon)$, $T_1/\rho_0 c_1^2 e = O(\epsilon^2)$ while T/T_1 varies over the full range $(0, 1)$. Note that the expressions (2.31)–(2.33) satisfy equations (2.3) and (2.5) with

$$\mu = 0 \quad \text{and} \quad \nu = -2\left(\epsilon\frac{\rho_1 c_1}{\rho_0}\right)^{-\frac{1}{2}}. \tag{2.34}$$

Outside the layer where the relations (2.31)–(2.33) hold, where $\epsilon \ll c/c_1 < 1$, as c/c_1 increases, $T/T_1 \equiv 1$ but $(T_1/\rho_0 c_1^2)\,e$ increases in the range $(0, \infty)$ while $(\rho_0 c_1/T_1)\,A$ decreases in the range $(\infty, 0)$. Typical responses of ideally soft materials as $M \to 0$ are depicted in figure 3.

2.3. *Stress–strain relations for non-ideally soft materials*

A typical stress–strain relation for a non-ideally soft material is depicted in figure 2. The sound speed A decreases monotonically with either increasing or decreasing e from A_0 at $e = 0$ to some limiting value $A_\infty < A_0$. The parameters (μ, ν) can be chosen so that, in addition to A_0, A_∞ and T_I (the value of T at which the two limiting tangents intersect) take specified values. T_1 is a measure of the rate at which A varies with T.

The variation of A with c in a non-ideal material is described by the single relation

$$A/A_0 = M\coth^2[\eta_0 + \eta_1(c/A_0)], \tag{2.35}$$

where $$\mu A_0^{\frac{1}{2}} = 2\eta_1 M^{\frac{1}{2}}, \quad \nu A_0^{\frac{3}{2}} = -2\eta_1 M^{\frac{1}{2}} \quad \text{and} \quad 0 \leqslant M \leqslant 1. \tag{2.36}$$

In equation (2.35), $\eta_1(c/A_0)$ can vary in the range $(0, \infty)$ and

$$\text{as} \quad \eta_1(c/A_0) \to \infty, \quad A \to A_\infty = MA_0. \tag{2.37}$$

The relation (2.35) can be used to describe the behaviour of many polycrystalline materials, such as aluminium and copper, during uniaxial compression. The stress–strain relations of such materials are usually fitted, with reasonable accuracy, by simple power laws over most of the range of interest. If (T_M, e_M) denote the maximum compressive stress and strain that occur during the deformation, these laws can be written

$$T_E/T_M = (e_E/e_M)^n, \tag{2.38}$$

where the exponent n lies somewhere in the range $(0, 1)$. This corresponds to

$$A_E/A_M = (c/c_M)^{(n-1)/(n+1)}, \tag{2.39}$$

where $$A_M = \left(n\frac{T_M}{\rho_0 e_M}\right)^{\frac{1}{2}} \quad \text{and} \quad c_M = \frac{2}{n+1}e_M A_M. \tag{2.40}$$

TABLE 3. THE VALUES OF η_0, η_1 AND M, TOGETHER WITH THE CORRESPONDING VALUES OF μ, ν AND A_M/A_0, FOR WHICH THE $A/A_0 = M \coth^2 [\eta_0 + \eta_1 (c/A_0)]$ BEST FITS THE EXPERIMENTAL LAW $A_E/A_M = (c/c_M)^{(n-1)/(n+1)}$ OVER THE RANGE $0.1 \leqslant c/c_M \leqslant 1$. THE RELATIVE ERROR IN T/T_M AND e/e_M IS LESS THAN 1 %

parameters	power $n = 1$	$n = \frac{1}{2}$	$n = \frac{1}{3}$	$n = \frac{1}{4}$
M	1	0.3570	0.2081	0.1487
η_0	∞	0.6892	0.4925	0.4066
$(c_M/A_0)\ \eta_1$	0	1.2601	1.2831	1.2907
A_M/A_0	1	0.3802	0.2262	0.1633
$A_0^{-\frac{1}{2}}\lvert c_M\rvert \mu$	-0	-1.5057	-1.1706	-0.9952
$A_0^{\frac{1}{2}}\lvert c_M\rvert \nu$	0	4.2180	5.6258	6.6951

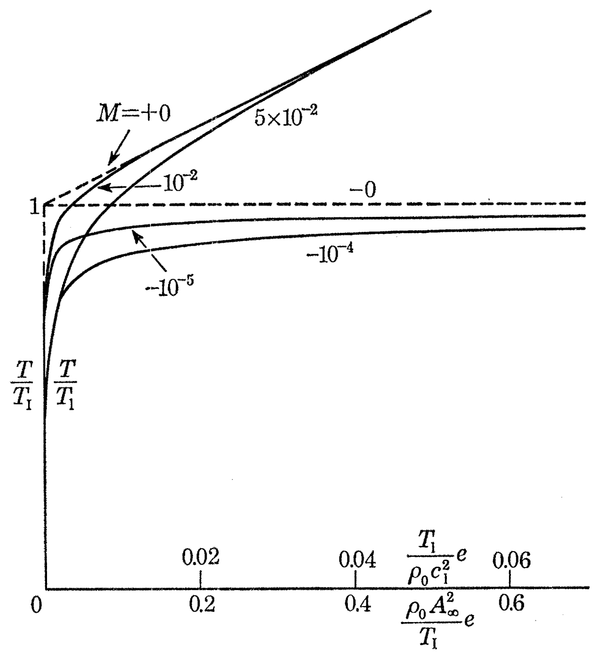


FIGURE 3. Typical stress–strain relations for rigid–elastic and rigid–plastic materials. The rigid–elastic response is obtained as $M \to 0$ from values > 0. For these materials both T_I and A_∞ can be specified. The rigid–plastic response is obtained as $M \to 0$ from values < 0. For these materials T_1 and c_1 can be specified: c_1 is the velocity with which the end of a bar will move when loaded by the limiting traction T_1.

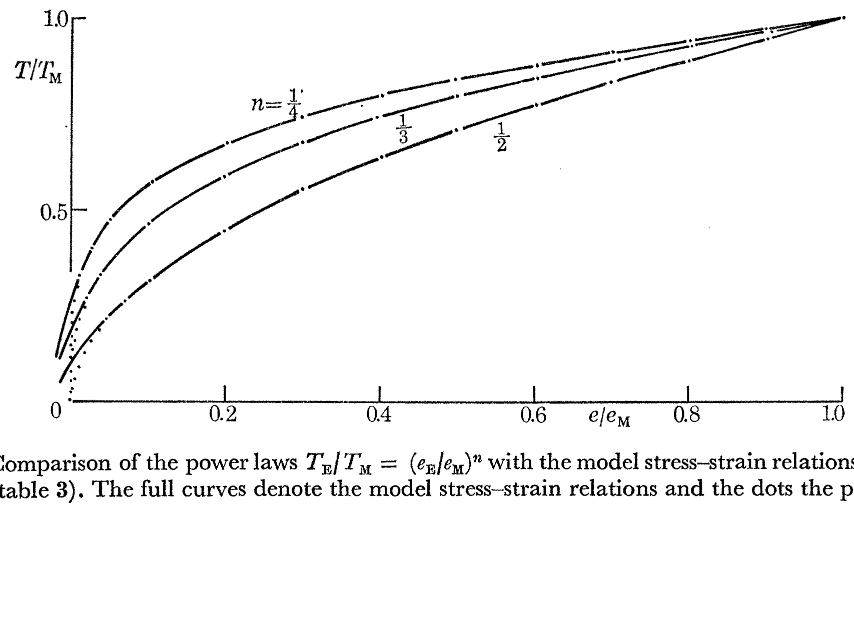


FIGURE 4. Comparison of the power laws $T_E/T_M = (e_E/e_M)^n$ with the model stress–strain relations when $n = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ (see table 3). The full curves denote the model stress–strain relations and the dots the power laws.

Table 3 displays the values of M, η_0 and η_1 for which the expression (2.35) 'best' approximates the expression (2.39) over the range $0.1 \leqslant c/c_M \leqslant 1$ when $n = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$. The corresponding stress–strain relations are compared in figure 4. The relative error over this range is less than 1 %. Of course, the variation (2.35) cannot be used to approximate the singular behaviour described by equation (2.39) as $c/c_M \to 0$. Equation (2.39) predicts that $A_E/A_M \to \infty$ as $c/c_M \to 0$: equation (2.35) predicts that A/A_M takes the values A_0/A_M displayed in table 3. In spite of this, the model stress–strain law provides an excellent fit to the law (2.38) over 97 % of the range of variation of e. In practice, of course, it is just as good a fit to the experimental data.

Some materials, such as yarns in tension, behave essentially rigid for $T < T_I$ and as Hookean materials for $T > T_I$. This response is described by equations (2.3) and (2.35) as

$$\tanh^2 \eta_0 = M \to 0, \quad MA_0 \to A_\infty \quad \text{and} \quad M^{\frac{3}{2}}\eta_1 \to \frac{\rho_0 A_\infty^2}{T_I}. \tag{2.41}$$

In this limit, the dominant behaviour is described by the equations

$$\frac{A}{A_\infty} = \left[M^{\frac{1}{2}} + \tanh\left(M^{-\frac{1}{2}}\frac{\rho_0 A_\infty}{T_I}c\right)\right]^{-2}, \tag{2.42}$$

$$\frac{T}{T_I} = \left[1 + M^{\frac{1}{2}}\coth\left(M^{-\frac{1}{2}}\frac{\rho_0 A_\infty}{T_I}c\right)\right]^{-1} + \frac{\rho_0 A_\infty}{T_I}c \tag{2.43}$$

and

$$\frac{\rho_0 A_\infty^2}{T_I} = \frac{\rho_0 A_\infty}{T_I}c - (1-M)\left[1 + M^{-\frac{1}{2}}\coth\left(M^{-\frac{1}{2}}\frac{\rho_0 A_\infty}{T_I}c\right)\right]^{-1}. \tag{2.44}$$

The almost rigid behaviour is obtained when

$$\frac{\rho_0 A_\infty}{T_I}c = O(M).$$

In this layer, if

$$\frac{\rho_0 A_\infty}{T_I}c = M\eta, \tag{2.45}$$

equations (2.42)–(2.44) predict that

$$\frac{A}{A_\infty} = M^{-1}(1+\eta)^{-2}, \quad \frac{T}{T_I} = \frac{\eta}{1+\eta} \quad \text{and} \quad \frac{\rho_0 A_\infty^2}{T_I}e = M^2(\eta + \eta^2 + \tfrac{1}{3}\eta^3). \tag{2.46}$$

Accordingly, as T/T_I varies in the range (0, 1), $(\rho_0 A_\infty^2/T_I)\, e = O(M^2)$. The Hookean behaviour is obtained where $(\rho_0 A_\infty/T_I)\, c = O(1)$ as $M \to 0$. Then, equations (2.42)–(2.44) predict that

$$\frac{A}{A_\infty} = 1, \quad \frac{T}{T_I} = 1 + \frac{\rho_0 A_\infty}{T_I}c \quad \text{and} \quad e = \frac{c}{A_\infty}. \tag{2.47}$$

If c is eliminated, these last two equations imply that

$$T = T_I + \rho_0 A_\infty^2 e. \tag{2.48}$$

Typical responses of non-ideal materials as $M \to 0$ are depicted in figure 3.

3. The initial deformation of an impulsively loaded elastic slab

A conceptually simple example of the deformations analysed in this paper occurs when a slab (panel, bar, string) of elastic material is suddenly loaded at one of its boundaries, $X = D$, by a normal traction which is then held constant for some time. During the ensuing deformation the other boundary, $X = 0$, is held rigid. There are two separate cases to consider. The first is when the material softens for the applied load, so that the load is compressive when the material softens in compression and tensile when the material softens in tension. The second case, which is a little more difficult to analyse, is when the material hardens for the applied load.

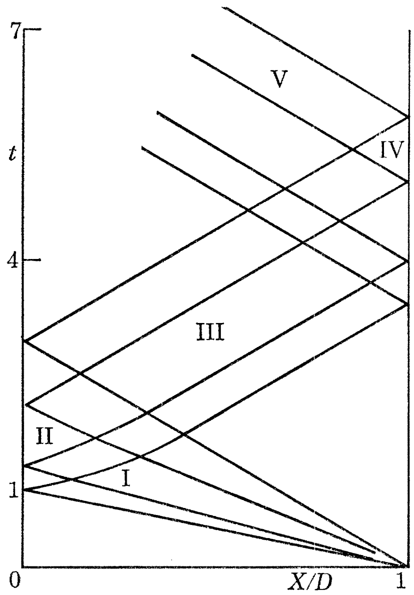


FIGURE 5

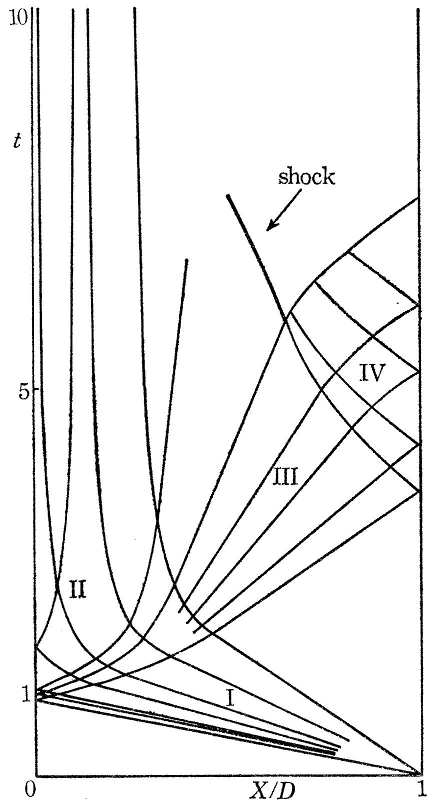


FIGURE 6

FIGURE 5. The initial wave pattern produced in a non-ideally soft material ($M = 0.36$) when a load is suddenly applied and then is held fixed at $X = D$. During the deformation the boundary $X = 0$ is rigid.

FIGURE 6. The initial wave pattern produced in an ideally soft material ($M = 1.1$) when a load is suddenly applied and then is held fixed at $X = D$. During the deformation the boundary $X = 0$ is rigid. The applied traction is large enough for the material to yield at $X = 0$. After its reflexion from $X = D$ the wave focuses and a shock forms.

When the material softens the reference state R is taken as the state the material is in before the application of the traction T_a. Then, a centred wave is generated at $X = D$ at the instant the load is applied. This moves with constant speed A_0 towards the rigid boundary $X = 0$, where it is completely reflected. Typical examples of the resulting wave patterns in non-ideal and ideal materials are shown in figures 5 and 6. As the centred wave moves towards $X = 0$ it traverses two distinct regions. In region I it is a simple wave. During the passage of this wave the material softens until $T = T_a$. This is accompanied by a decrease in A from A_0 to A_a. Thereafter, the material remains in a constant state until the arrival of the front of the wave reflected from $X = 0$. In region II, the first interaction region, the centred wave interacts with the wave reflected from $X = 0$. No shocks form because the effect of a rigid boundary is to further soften the material. After traversing the interaction region the reflected wave emerges as a simple wave into region III, where it moves with constant speed A_a towards $X = D$. During its passage the material continues to soften. When this wave approaches $X = D$ it too begins to interact with the wave reflected from this boundary. The deformation in this second interaction region (region IV) depends on conditions at this boundary. In the somewhat idealized situation when T continues to be held constant the effect of the wave reflected from $X = D$ is to begin to harden the material. Then, in general, one of two things happens: either the material stops behaving elastically or the reflected wave begins to focus. In the latter case a shock may form.

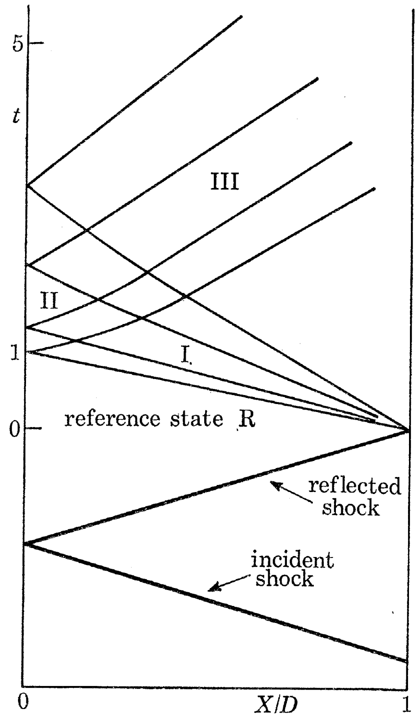


FIGURE 7. The initial wave pattern produced in a material that hardens when a load is applied at $X = D$. After the load is applied it is held fixed while the boundary $X = 0$ remains rigid. A shock forms at the instant the load is applied and moves with constant speed towards $X = 0$ where it is reflected as a constant strength shock. This shock is reflected from $X = D$ at $t = 0$ as a centred wave. The reference state R is that induced after the passage of the reflected shock.

If the material hardens when the load is applied, a shock forms immediately and moves with constant speeds towards the rigid boundary $X = 0$ (see figure 7). Behind this shock the material is in a uniform state with $T = T_a$ and $u = U$. When this shock reaches $X = 0$ it is reflected as a constant strength shock that moves with constant speed towards the loaded boundary $X = D$. Between this shock and $X = 0$ the material is in a uniform state with $u = 0$ and $T = T_0$. The constants U and T_0 can be computed in terms of T_a and conditions in the slab before the application of the load by using the usual shock relations and the Hugoniot curve for the material (see, for example, Courant & Friedrichs 1948). When the shock that is reflected from $X = 0$ reaches $X = D$ it is reflected as a centred wave across which the traction changes from T_0 to T_a. Consequently, *if the reference state* R *is taken as the state that is induced after the passage of the shock reflected from* $X = 0$, *so that* T_a *is measured relative to* T_0 *and the strain is computed relative to this configuration, the analysis of the subsequent deformation is identical with that for a material that softens when the load* T_a *is applied.*

3.1. *Reflexion from an interface separating different elastic materials*

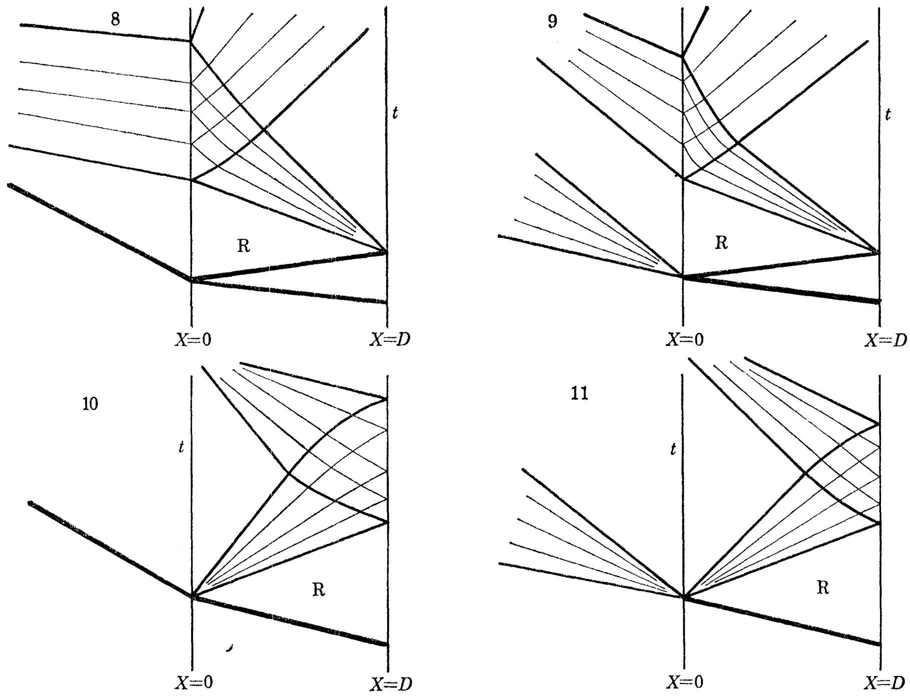
The analysis presented in this paper is valid for a rather more general situation than that described above. Instead of $X = 0$ being just a rigid boundary, it is taken as an interface with some other elastic material. At this interface part of the energy of the incident centred wave is reflected and part is transmitted. However, it is assumed that the energy which may be reflected from the other boundary of the surrounding material has no influence on the deformation at $X = 0$. The limiting cases of a perfectly rigid and perfectly free interface are given special attention.

When the material softens for the applied load, the deformation at $X = 0$ is continuous. Then, as was shown in part I, the wave transmitted into the surrounding material is a simple wave. Consequently, if $T_L = T_L(c_L)$ is the equation of state of the surrounding material, the condition that $c_L = u_L$ in the transmitted wave together with the conditions that $T_L = T$ and that $u_L = u$ at the interface implies that

$$\text{at} \quad X = 0, \quad T_L(u) = T(c). \tag{3.1}$$

Condition (3.1), with c and u given by equations (2.2) and (2.4), determines a relation

$$F = L(G) \quad \text{at} \quad X = 0. \tag{3.2}$$



FIGURES 8–11. The four possible wave patterns that can occur when an elastic slab is suddenly loaded at $X = D$ by a traction that generates a shock. The interface $X = 0$ separates the slab from some other elastic material.

When the material hardens for the applied load, the incident shock can be reflected from $X = 0$ either as a shock or as a centred wave. In either case the transmitted wave can be a shock or a centred wave. The four possible wave patterns at $X = 0$ are depicted in figures 8–11. The strengths of the transmitted and reflected waves can be calculated by the same procedure as used by Courant & Friedrichs (1948) to obtain the strengths and natures of the waves that are produced when a constant strength shock travelling through a gas reaches a contact discontinuity. The arguments are straightforward, even though the algebra is rather messy: they will not be presented here.

When the wave reflected from $X = 0$ is a shock, as is depicted in figures 8 and 9, the material is in a uniform state after its passage with $u = u_0$ and $T = T_0$. If this state is taken as the reference state R, the analysis of the deformation after the shock is reflected from $X = D$ as a centred wave is almost identical with that for a material that softens when the load is applied. The only difference is that $u = u_0$ ahead of the centred wave. The two problems are identical if a velocity $-u_0$ is imposed on the system.

When the wave reflected from $X = 0$ is a centred wave, as it is when $X = 0$ is a perfectly free boundary, the start induced by the passage of the incident shock is taken as the reference state R (see figures 10 and 11). Thus, in the measurement scale adopted, the boundary $X = D$ acts as a perfectly free boundary at the arrival of the wave centred at $X = 0$. Consequently, a description of the deformation during this reflexion can readily be obtained from an analysis of the deformation that occurs when a wave centred at $X = D$ is reflected from a perfectly free interface at $X = 0$.

3.2. *Shock loading*

The solution of the problem outlined in §3.1 can be used to describe the early stages of many practically important deformations. One example is the deformation produced in a slab of elastic material when a constant strength shock wave travelling through an adjacent material arrives at their common interface. During the subsequent deformation energy is radiated across the other boundary of the slab to some adjacent material. For example, the slab could be a panel surrounded on both sides by air, or it could be an element of some composite material.

Suppose that the incident shock travels through the material to the right of the slab. When it reaches $X = D$ either a centred wave or a shock wave is transmitted into the slab and either a centred wave or a shock wave is reflected. In all cases the traction at $X = D$ changes discontinuously at the arrival of the shock and remains constant until the arrival of the wave reflected from $X = 0$. The analysis of the deformation during this first reflexion is identical with that for the problem described in §3.1. Only at the arrival of the reflected wave at $X = D$ do the characters of the two deformations begin to differ. Now, energy is radiated across the interface $X = D$. Note though that if a shock wave is transmitted into the slab and if this is reflected from $X = 0$ as a centred wave (as is illustrated in figures 10 and 11) the deformation during the first reflexion from $X = D$ can also be described by our analysis. For when this wave reaches $X = 0$ a simple wave is transmitted into the adjacent material. Consequently, by using the same argument as that used in §3.1, during the reflexion of this wave

$$\text{at} \quad X = D, \quad G = R(F), \tag{3.3}$$

where the function $R(F)$ is determined by the equations of state of the two materials separated by the interface $X = D$. On the other hand if the transmitted wave is a shock and if this is reflected as a shock from $X = 0$ (as illustrated in figures 8 and 9) any one of the four possible wave configurations can occur when this shock reaches $X = D$. For two of these the wave reflected from $X = D$ is a centred wave and the subsequent deformation is similar to that produced when the wave that is first transmitted into the slab is a centred wave. For the two other possible wave configurations the wave reflected from $X = 0$ is again a shock and the subsequent deformation is similar to that produced when the transmitted wave was a shock.

It should be pointed out that in many materials the stress–strain relation differs from particle to particle after the passage of a shock of variable strength. This is caused either by the production of strong entropy gradients (as in gases) or by the basic hysteretic structure of the material (as in soils). In general this induced stratification of the material greatly complicates the analysis of its subsequent deformation. Here, though, since we only deal with constant strength shocks these difficulties do not occur although our analysis is ony applicable to hysteretic materials as long as the material continues to soften.

3.3. *Impact loading*

An analysis of the problem outlined in § 3.1 can also be used to describe the early stages of the deformation that occurs when a bar, or a slab, moving with a constant velocity $-u_a > 0$ impacts a rigid wall. Then, referred to a coordinate system moving with the bar, at the moment of impact the velocity of the end of the bar changes discontinuously from zero to u_a. It stays at this value while the end of the bar remains in contact with the wall. If the material softens in compression, a centred wave is generated at $X = D$ at the moment of impact. Although this wave is reflected as a continuous wave from $X = 0$ it may, subsequently, focus and form a shock before reaching the boundary $X = 0$. If, however, the material hardens in compression a shock is generated at $X = D$ at the moment of impact. This is reflected from $X = 0$ as a centred wave. This centred wave is, in turn, reflected from the rigid boundary $X = D$ as a continuous wave.

Obviously a solution of the general problem outlined in § 3.1 can also be used to analyse a slightly more general impact problem than that described above. The 'hitter' bar can be a composite of two materials separated by the interface $X = 0$ and the impacted material can also be elastic.

4. The interaction of a centred wave with any wave travelling in the opposite direction

In this section we analyse the problem described in § 3.1. This is done by obtaining a representation (equations (4.8) and (4.9)) that describes the interaction of a wave centred at $X = D$ at $t = 0$ with any wave travelling in the opposite direction. Then, in subsequent sections we show how to determine the arbitrary functions occurring in this representation when the opposite travelling wave is the wave that is reflected when the centred wave arrives at an interface with some other elastic material.

We suppose that at $t = 0$ the bar is in equilibrium in its reference state R where all the state variables (u, c, T, e) are zero while $A = A_0$. Then, at $t = 0$ the traction at $X = D$ changes discontinuously to T_a. This generates a centred wave at $X = D$ that moves with speed A_0 towards the interface $X = 0$. There the wave is partly reflected and partly transmitted. The centred wave traverses two distinct regions. In region I, neighbouring $X = D$, $F \equiv 0$ and the wave is a centred simple wave. During its passage, at any X, T changes monotonically in time from zero to T_a. Therefore, $T \equiv T_a$ until the arrival of the wave that is reflected from $X = 0$. As T changes from zero to T_a, the state variables u, c and G, which are equal, change from zero to c_a; e changes to e_a; and A decreases from A_0 to A_a. The constants c_a, e_a and A_a can be computed in terms of T_a by inserting the expressions (2.13), (2.15) and (2.35) for $\hat{A}(c)$ in the relations (2.3). When the applied load is tensile $(T_a, c_a) > 0$; when the applied load is compressive $(T_a, c_a) < 0$. In region II, the first interaction region, the centred wave interacts with the wave reflected from the interface $X = 0$. Here F is not zero but must be calculated from the fact that at $X = 0$, u and c are related by equation (3.1).

In part I it was shown that as the centred wave traverses both region I and II, equation (2.7) implies that

$$2A^{\frac{1}{2}}\frac{\partial t}{\partial \beta}+[\mu t+\nu(1-X)]\,G'(\beta) = 0.\dagger \tag{4.1}$$

If we use the fact, which follows from equations (2.2) and (2.5), that

$$G'(\beta) = \frac{\partial c}{\partial \beta} = A^{-\frac{1}{2}}(\mu+\nu A)^{-1}\frac{\partial A}{\partial \beta}, \tag{4.2}$$

together with the fact, which follows from the first of equations (2.1), that

$$\frac{\partial}{\partial \beta}[\mu t+\nu(1-X)] = [\mu-\nu A]\frac{\partial t}{\partial \beta}, \tag{4.3}$$

equation (4.1) implies that

$$2A\left(\frac{\mu+\nu A}{\mu-\nu A}\right)\frac{\partial}{\partial \beta}[\mu t+\nu(1-X)]+\frac{\partial A}{\partial \beta}[\mu t+\nu(1-X)] = 0. \tag{4.4}$$

This equation can be written $$\frac{\partial}{\partial \beta}\left([\mu t+\nu(1-X)]\frac{A^{\frac{1}{2}}}{\mu+\nu A}\right) = 0, \tag{4.5}$$

which integrates to give $$\mu t+\nu(1-X) = \phi(\alpha)\,A^{-\frac{1}{2}}(\mu+\nu A), \tag{4.6}$$

where $\phi(\alpha)$ is a 'constant' of integration. When the expression (4.6) for $[\mu t+\nu(1-X)]$ is inserted in equation (4.3) this yields the equation

$$\frac{\partial t}{\partial \beta} = \phi(\alpha)\frac{\partial A^{-\frac{1}{2}}}{\partial \beta}. \tag{4.7}$$

Equations (4.6) and (4.7) imply that in the centred wave t and X can always be expressed in terms of A and α by relations of the form

$$t = \phi(\alpha)\,A^{-\frac{1}{2}}+\tau(\alpha) \tag{4.8}$$

and $$1-X = \phi(\alpha)\,A^{\frac{1}{2}}+M\tau(\alpha), \tag{4.9}$$

where $\tau(\alpha)$ is a 'constant' of integration for equation (4.7). The material parameter

$$M = -\mu/\nu \quad (= -\mu/A_0\nu \text{ in dimensional variables}). \tag{4.10}$$

Once the functions $\phi(\alpha)$ and $\tau(\alpha)$ have been determined, equations (4.8) and (4.9) determine $A(x,t)$. In particular, to calculate the variation of A with t at any specified particle note that at constant X equation (4.9) determines A as an *explicit* function of the characteristic parameter α. When this expression for A is inserted, equation (4.8) then determines t as an explicit function of α. Consequently, the variation of A with t is easily obtained. Similarly, the variation of A with X at fixed t follows immediately from equations (4.8) and (4.9). Once $A(X,t)$ has been determined, the variations of c, T and e follow from equations (2.3), (2.13), (2.15) and (2.35). Note that the trajectories of constant stress and strain can be obtained from equations (4.8) and (4.9) by holding A fixed and varying α.

† In this equation, and from now on, distance is measured in units of D, time in units of D/A_0, and velocity in units of A_0. Also, stress is measured in units of $\rho_0 A_0^2$ and the material constants μ and ν in units of $A_0^{-\frac{1}{2}}$ and $A_0^{-\frac{3}{2}}$ respectively. This measurementt scale is adopted to avoid messy algebraic expressions.

Equations (4.8) and (4.9) also determine the trajectories of the α-characteristics. For if A is eliminated from these equations, they yield the relation

$$(1-X-M\tau)(t-\tau) = \phi^2. \tag{4.11}$$

Since τ and ϕ are constant when α is constant, equation (4.11) determines the relation between X and t at any α-characteristic: their graphs are always hyperbolas in the (X, t) plane.

It remains to determine $u(X, t)$. To do this first note that the functions $F(\alpha)$, $\phi(\alpha)$ and $\tau(\alpha)$ are not independent but are related by the condition that

$$\mathrm{d}\tau/\mathrm{d}\alpha = \nu\phi\,\mathrm{d}F/\mathrm{d}\alpha. \tag{4.12}$$

This follows when the expressions (4.8) and (4.9) for t and X are inserted in the second of equations (2.1) if the facts that $A(c)$ satisfies equation (2.5) and that c is given by equation (2.2) are used. Then, use the fact, which follows from equations (2.2) and (2.4), that u can be expressed in terms of A and α as

$$u = \hat{c}(A) - 2F(\alpha), \tag{4.13}$$

where the function $c = \hat{c}(A)$ is obtained from relation (2.13) when $1 \leqslant M \leqslant \infty$, from relation (2.15) when $-\infty \leqslant M \leqslant 0$, and from relation (2.35) when $0 \leqslant M \leqslant 1$. Equations (4.8), (4.9) and (4.13) determine $u(X, t)$ once the functions ϕ, τ and F are known.

The trajectories of the β-characteristics can be calculated from equation (2.2). This states that

$$\text{at constant } \beta, \quad \hat{c}(A) - F(\alpha) = G, \quad \text{is constant.} \tag{4.14}$$

5. Incident wave

To calculate ϕ and τ in region I, first note that in the reference state R where $A = 1$ ($= A_0$ in dimensional variables) equations (2.1) integrate, subject to conditions (2.8) and (2.9), to give

$$\alpha = t - X \quad \text{and} \quad \beta = t + X - 1. \tag{5.1}$$

Consequently, at the front of the centred wave, where $\beta = 0$,

$$t = \tfrac{1}{2}(1+\alpha), \quad X = \tfrac{1}{2}(1-\alpha), \quad A = 1 \quad \text{and} \quad F = 0 \quad \text{for} \quad -1 \leqslant \alpha \leqslant 1. \tag{5.2}$$

When the information (5.2) is inserted in equations (4.8) and (4.9) these yield the results that

$$\phi = \tfrac{1}{2}(1+\alpha) \quad \text{and} \quad \tau = 0 \quad \text{for} \quad -1 \leqslant \alpha \leqslant 1. \tag{5.3}$$

Equations (4.8) and (4.9) then imply that

$$A = (1-X)/t \quad \text{for} \quad A_a \leqslant A \leqslant 1, \tag{5.4}$$

and that

$$\alpha = 2[(1-X)\,t]^{\frac{1}{2}} - 1 \quad \text{for} \quad -1 \leqslant \alpha \leqslant 1. \tag{5.5}$$

Since $F = 0$, equations (4.13) and (5.4) imply that

$$u = c = \hat{c}[(1-X)/T]. \tag{5.6}$$

Also, according to equation (4.14) and (5.6), the β-characteristics are the rays

$$(1-X)/t = \text{constant}. \tag{5.7}$$

The solutions described by equations (5.4)–(5.7) are the well-known centred simple wave solutions.

6. The interaction of the incident centred wave and the wave reflected from a perfectly free interface

The incident centred wave remains a simple wave until it is met by the front of the wave reflected from $X = 0$. The trajectory of this front is given by equation (5.5) with $\alpha = 1$: it is

$$(1-X)\,t = 1. \tag{6.1}$$

At this front, equations (5.4) and (6.1) imply that

$$A = (1-X)^2. \tag{6.2}$$

Since the interaction region is completely traversed by the front when $A = A_a$, in equation (6.1)

$$0 \leqslant X \leqslant 1-A_a^{\frac{1}{2}} \quad \text{and} \quad 1 \leqslant t \leqslant A_a^{-\frac{1}{2}}. \tag{6.3}$$

6.1. *Perfectly free interface*

As a first, comparatively simple, illustration of how to use the results established in §4 to calculate the deformation in region II we consider the case when $X = 0$ is a perfectly free interface. Then,

$$\text{at} \quad X = 0, \quad \text{where} \quad t = \alpha, \quad T = 0, \quad A = 1 \quad \text{and} \quad c = 0. \tag{6.4}$$

When the information (6.4) is inserted in equations (4.8) and (4.9) these imply that

$$\tau = (\alpha-1)/(1-M) \quad \text{and that} \quad \phi = (1-M\alpha)/(1-M). \tag{6.5}$$

Conditions (4.12) and (6.5), together with the result that $F = 0$ when $\alpha = 1$, then imply that

$$F = \mu^{-1}\ln\phi. \tag{6.6}$$

The relations (6.5) hold for $1 \leqslant \alpha \leqslant \alpha_a$, where

$$\alpha_a = 1+\frac{1-M}{M}[1-\exp(-\mu c_a)]. \tag{6.7}$$

This follows from equations (6.5) and (6.6) and the fact that the back of the centred wave, at which $G = c_a$, reaches $X = 0$ when $F = -G = -c_a$.

When the expressions (6.5) are inserted in equations (4.8) and (4.9) they imply that

$$\phi = (1-Mt)\frac{A^{\frac{1}{2}}}{A^{\frac{1}{2}}-M} = \frac{X}{1-A^{\frac{1}{2}}}. \tag{6.8}$$

The second of these two relations yields the result that

$$A = \tfrac{1}{4}\{(1-\theta)+[(1-\theta)^2+4M\theta]^{\frac{1}{2}}\}^2 \quad \text{where} \quad \theta = X/(1-Mt). \tag{6.9}$$

These equations determine A as an explicit function of (X, t). To determine $u(X, t)$ use equations (4.13), (6.6) and (6.8): these yield

$$u = \hat{c}(A) - 2\mu^{-1}\ln\phi, \tag{6.10}$$

where $\phi(X, t)$ and $A(X, t)$ are given by equations (6.8) and (6.9).

Since ϕ is constant at any α-characteristic their trajectories can readily be determined from equations (6.8) and (6.9). To calculate the trajectories of the β-characteristics as they are refracted by the reflected α-wave use equations (4.14), (6.6) and (6.8). These imply that a constant β,

$$(1-Mt)\frac{A^{\frac{1}{2}}}{A^{\frac{1}{2}}-M}\exp\left[-\mu\hat{c}(A)\right] = \frac{X}{1-A^{\frac{1}{2}}}\exp\left[-\mu\hat{c}(SA)\right] = \text{constant}, \tag{6.11}$$

$$= \frac{1-M\bar{\beta}}{1-M} \quad \text{say}, \tag{6.12}$$

where $\bar{\beta}$ is the arrival time of the β-characteristic at $X = 0$. In particular, the trajectory of the back of the centred wave, which corresponds to $\bar{\beta} = \alpha_a$, is given parametrically by

$$1-Mt = \frac{A^{\frac{1}{2}}-M}{A^{\frac{1}{2}}}\exp\left[\mu(\hat{c}(A)-c_a)\right] \quad \text{and} \quad X = (1-A^{\frac{1}{2}})\exp\left[\mu(\hat{c}(A)-c_a)\right], \tag{6.13}$$

for $A_a \leqslant A \leqslant 1$.

According to equations (6.9) constant levels of θ, which correspond to constant levels of A, T, c and e, propagate with constant speed $|M\theta|$. The direction of propagation is towards $X = 0$ (like in the incident wave) for a non-ideal material and away from $X = 0$ for an ideal material. This θ-wave is centred at $(X, t) = (0, M^{-1})$. For ideal materials $t = M^{-1} \leqslant 1$ – the instant the front of the incident wave arrives at $X = 0$. For non-ideal materials $t = M^{-1} \geqslant \alpha_a$ – the instant the back of the incident wave arrives at $X = 0$. It should be noted that equations (6.8) and (6.10) imply that constant levels of u do not usually coincide with constant levels of θ.

6.2. *Shock formation*

The main effect of the wave that is reflected from a free interface is to harden, or unload, the material. This causes one of two things to happen: the material either stops behaving elastically or the reflected wave focuses and a shock may form. The results derived in § (6.1), and in the remainder of the main body of this paper, are only valid for materials that continue to behave elastically on unloading.

Although the reflected α-wave can never focus rapidly enough for a shock to form in region II when the material is non-ideal, if T_a/T_i is sufficiently large a shock can form in an ideal material. This is best seen by noting that at any constant X

$$\frac{DA}{Dt} = \frac{M\theta}{1-Mt}\bar{A}'(\theta), \tag{6.14}$$

where $\bar{A}(\theta)$ is given by equation (6.9). Accordingly, a shock forms when $\bar{A}'(\theta)$ is unbounded. This happens when $\theta = \theta_s$ and $A = A_s$, where

$$(1-\theta_s)^2 + 4M\theta_s = 0 \quad \text{and} \quad A_s = -M\theta_s. \tag{6.15}$$

The largest value of A_s lying in the range $(A_a, 1)$ determines the shock that forms first: it is

$$A_s = M^2[1-(1-M^{-1})^{\frac{1}{2}}]^2. \tag{6.16}$$

Since A_s is always imaginary when M lies in the range (0, 1), no shock can form in a non-ideal material in region II. However, if T_a/T_i is sufficiently large for A to attain the value $A_s(M)$ a shock will form in an ideal material. This happens whenever

$$T_a/T_i > S_2(M), \tag{6.17}$$

where the function $S_2(M)$ is graphed in figure 12. $S_2(M)$ is the value of T/T_i corresponding to $A = A_s(M)$. Note that the criterion for shock formation only depends on the material parameter M. When the condition (6.17) is satisfied a shock forms at the front of the reflected wave. This follows from the second of equations (6.9) and equations (6.14). These imply that the ray $\theta = \theta_s$ does not actually intersect the interaction region but is tangent to the front (6.1) at the point (X_{s_2}, t_{s_2}), where

$$X_{s_2} = 1 - A_s^{\frac{1}{2}} \quad \text{and} \quad t_{s_2} = A_s^{-\frac{1}{2}}. \tag{6.18}$$

Once a shock forms it produces a reflected wave that moves towards the free interface $X = 0$. The front of this wave is the β-characteristic that passes through the point (X_{s_2}, t_{s_2}) when $A = A_s$. The description of the deformation given in § 6.1 is only strictly valid at a particle until the time that this front arrives. However, in practice, it may provide a good approximation as long as the wave reflected from the shock remains weak.

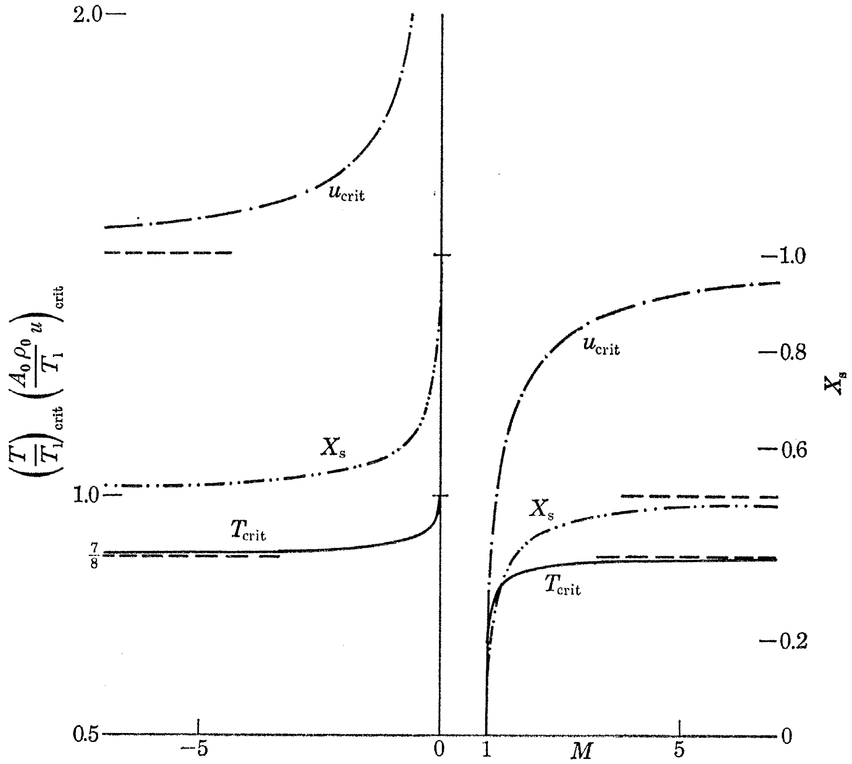


FIGURE 12. The variations with M of the least value of T_a/T_1, and the equivalent value of $(\rho_0 A_0/T_1)\,u_a$, for which a shock will form in region II during the reflexion of a centred wave from a free surface. The shock forms at the front of the reflected wave at $X = X_s(M)$.

6.3. *Limiting cases*

(i) *Hookean material* ($M \to 1-$)

When the material response is linear any discontinuous change in T at $X = 1$ is not smoothed by amplitude dispersion but propagates as a discontinuity towards $X = 0$, where it is reflected as a discontinuity. In order to illustrate the effect of small deviations from linear response, and at the same time to relate conditions across the incident and reflected discontinuities, it is instructive to consider the limiting behaviour as $M \to 1$ from values < 1. According to the results derived in part I (see I, (9.15)), in this limit the incident centred wave is a fan in which

$$A = (1 - X)/t = 1 + O(1 - M). \tag{6.19}$$

In this fan, to a first approximation,

$$u = c = e = T = -T_l \ln\left[1 - \frac{1}{1-M}(1-A)\right] \quad \text{for} \quad 0 \leqslant T/T_a \leqslant 1. \tag{6.20}$$

In the interaction region $X = O[(1-M)]$ and $t = 1 + O[(1-M)]$. Consequently, it is convenient to work with the variables

$$\bar{X} = (1-M)^{-1}X \quad \text{and} \quad \bar{t} = (1-M)^{-1}(t-1). \tag{6.21}$$

Then, conditions (6.1)–(6.2), (6.13) and (6.19)–(6.20) imply that the interaction region is bounded by the trajectories

$$\bar{t} - \bar{X} = 0 \quad \text{and} \quad \bar{t} + \bar{X} = 1 - \exp(T_a/T_l). \tag{6.22}$$

In this region conditions (6.8) and (6.9) imply that

$$A = 1 - (1-M)\frac{2\bar{X}}{1+\bar{X}-\bar{t}} \quad \text{and} \quad \phi = 1 + \bar{X} - \bar{t} \tag{6.23}$$

so that, by conditions (6.20),
$$T = e = -T_l \ln\left[\frac{1-\bar{t}-\bar{X}}{1-\bar{t}+\bar{X}}\right]. \tag{6.24}$$

Also, conditions (6.10) and (6.23), with $\hat{c}(A)$ determined from conditions (6.20), yields

$$u = -T_l \ln[(1-\bar{t})^2 - \bar{X}^2]. \tag{6.25}$$

According to equation (6.22) the incident wave is completely reflected from the free interface $\bar{X} = 0$ when
$$\bar{t} = 1 - \exp(-T_a/T_l). \tag{6.26}$$

By this time, condition (6.25) implies that the velocity of the interface

$$u = 2T_a, \tag{6.27}$$

which, according to equation (6.20), is twice its value in the incident wave. Note that although the result (6.23) predicts that A increases with t at any fixed X, the reflected wave can never focus rapidly enough for a shock to form in the interaction region.

(ii) *Perfectly elastic–perfectly plastic response* ($M \to 1+$)

To discuss the limit as $M \to 1$ from values > 1 it is convenient to introduce the parameter

$$\eta_0 = -\tfrac{1}{2}\ln(M-1). \tag{6.28}$$

Then, in the incident centred wave, conditions (2.25) and (5.4) imply that

$$\text{as} \quad \eta_0 \to \infty, \quad A = (1-X)/t = \tanh^2[\eta_0(1-T/T_l)]/\tanh^2\eta_0 \quad \text{for} \quad 0 \leqslant T/T_l \leqslant T_a/T_l \leqslant 1. \tag{6.29}$$

In addition,
$$c = u = T, \tag{6.30}$$

while e is given in terms of T by equation (2.26). Conditions (6.29) and (6.30) imply that T/T_l and u/T_l can change from zero to any value $\sigma < 1$ in a fan where

$$A = (1-X)/t = 1 + O[(M-1)^{1-\sigma}]. \tag{6.31}$$

In this part of the fan
$$T = e = c = -T_l \ln\left(1 + \frac{1}{4}\frac{1-A}{M-1}\right)\Big/\ln(M-1). \tag{6.32}$$

However, if the applied traction T_a approaches the limiting value T_l, A varies over the full range $(1, 0)$ in the incident fan and the full description (6.29) must be used.

Since A_a can take the limiting value zero, conditions (6.1)–(6.3) imply that the interaction region is not necessarily limited to some small neighbourhood of $X = 0$ but can fill the whole slab. However, conditions (6.16), (6.18) and (6.29) imply that the wave reflected from a free interface must always focus and form a shock if $T_a/T_l \geqslant \frac{1}{2}$. Since the shock forms at the front of the reflected wave at the point

$$X_{2s} = (M-1)^{\frac{1}{2}} - (M-1) + O[(M-1)^{\frac{3}{2}}] \quad \text{and} \quad t_{s_2} = 1 + (M-1)^{\frac{1}{2}} + O[(M-1)^{\frac{1}{2}}] \quad (6.33)$$

only that part of the interaction region in which X and $t-1$ are $O[(M-1)^{\frac{1}{2}}]$ is unaffected by the shock.

A careful study of the representations (6.8)–(6.10) shows that in that part of the interaction region where X and $t-1$ are $O[(M-1)^{\frac{1}{2}}]$ the statements (6.23) for A and ϕ still continue to hold. T, e and c can then be determined as explicit functions of $\bar{X}$ and $\bar{t}$ from conditions (6.32), and u from the condition that

$$u = T_l \ln\left[\frac{X^2}{(1-A)^2}\left(4+\frac{1-A}{M-1}\right)\right]\Big/\ln(M-1). \quad (6.34)$$

The result (6.34) follows from condition (6.10) and the fact that

$$\text{as} \quad M \to 1+, \quad \mu^{-1} \simeq T_l/\ln(M-1). \quad (6.35)$$

Although the representations (6.32) and (6.34) remain valid, in some neighbourhood of the point (6.33) where

$$X = O[(M-1)^{\frac{1}{2}}], \quad = (M-1)^{\frac{1}{2}}\bar{\bar{X}} \quad \text{say,} \quad \text{and} \quad t-1-X = O[(M-1)] = (M-1)(\bar{X}-\bar{t}), \quad (6.36)$$

the representation (6.23) for A is invalid. In this region, conditions (6.9) imply that

$$\theta = -1 + (M-1)^{\frac{1}{2}}\bar{\theta} \quad \text{where} \quad \bar{\theta} = (1+\bar{X}-\bar{t})/\bar{\bar{X}}, \quad (6.37)$$

and that

$$A = 1 + (M-1)^{\frac{1}{2}}[(\bar{\theta}^2-4)^{\frac{1}{2}} - \bar{\theta}] \quad \text{for} \quad 2 \leqslant \bar{\theta} < \infty. \quad (6.38)$$

Conditions (6.32) and (6.34), with A given by equations (6.37) and (6.38), determine T, e, c and u as explicit functions of X and t. The shock point (6.33) corresponds to $\bar{\theta} = 2$. The expression (6.23) for A is obtained from equations (6.36) and (6.38) in the limit as $\bar{\theta} \to \infty$ (holding $\bar{t}$ and $\bar{X}$ fixed but letting $\bar{\bar{X}} \to 0$).

(iii) *Rigid–elastic response* $(M \to 0+)$

To discuss the behaviour of a rigid–elastic material we suppose that†

$$T_a/T_I > 1 + |O(M^{\frac{1}{2}})|. \quad (6.39)$$

Then, it follows from conditions (2.24) and (2.43) that

$$A_a = O(M) \quad \text{so that} \quad A_a/A_\infty = O(1), \quad (6.40)$$

while

$$Mc_a \to T_a/T_I - 1. \quad (6.41)$$

It is best to work with the time measure

$$t^* = Mt \quad (6.42)$$

† This means that $1 + dM^{\frac{1}{2}} < T_a/T_I \leqslant \infty$ for some $d > 0$.

rather than with t. (t^* measures time in units of D/A_∞, t measures time in units of D/A_0.) On this scale the fronts of the incident and the reflected waves propagate with infinite speed. Consequently, they are represented by the single curve $t^* = 0$ in the (t^*, X) plane. Figure 13 depicts a typical interaction region in the (t^*, X) plane. It is bounded by the curves

$$X = 0, \quad 1-X-t^* = 0, \quad t^* = 0 \quad \text{and} \quad t^* = \alpha_{\text{a}}^*, \tag{6.43}$$

where
$$\alpha_{\text{a}}^* = M\alpha_{\text{a}} = 1-\exp\left[2(1-T_{\text{a}}/T_I)\right]. \tag{6.44}$$

The statement (6.44) follows from equation (6.7) and the fact that

$$\mu/M \to 2T_I^{-1} \quad \text{as} \quad M \to 0+, \tag{6.45}$$

which follows from equations (2.42)–(2.48). Figure 13 also depicts the limiting trajectories of the α and β characteristics and the trajectories of constant levels of T and e.

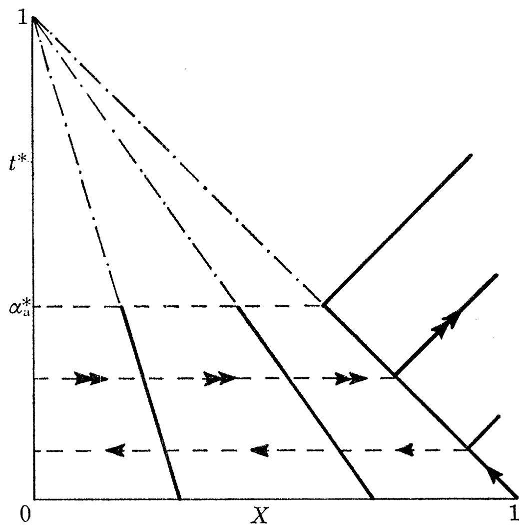


FIGURE 13. The reflexion of a centred wave from a free interface in a rigid–elastic material ($M \to 0+$). The interaction region is bounded by the curves $X = 0$, $1-X-t^* = 0$, $t^* = 0$ and $t^* = \alpha_{\text{a}}^*$ where $t^* = Mt$ and $\alpha_{\text{a}}^* = 1-\exp\left[2(1-T_{\text{a}}/T_I)\right]$. ——, The trajectories of constant stress; –◀–, β-characteristics; –▶▶–, α-characteristics.

Over the most of the interaction region, outside a fan centred at $(t^*, X) = (1, 0)$, where $\theta = 1-O(M^{\frac{1}{2}})$, the material is in a rigid state with $T/T_I < 1$. There, equations (6.9) predict that

$$A = (1-\theta)^2 = \left(\frac{1-X-t^*}{1-t^*}\right)^2. \tag{6.46}$$

It follows from the first and second of the equations (2.46) that

$$\frac{T}{T_I} = \theta = \frac{X}{1-t^*} \tag{6.47}$$

while, according to equations (2.46) and (6.46)

$$\frac{\rho_0 A_\infty^2}{T_I} e = \tfrac{1}{3}M^2\left[\left(\frac{1-t^*}{1-t^*-X}\right)^3 - 1\right]. \tag{6.48}$$

Also, according to equations (6.8), (6.9) and (6.45),

$$\frac{\rho_0 A_\infty}{T_I} u = -\ln(1-t^*), \tag{6.49}$$

so that the motion of the slab is independent of X. In equation (6.49), for ease of interpretation, A_∞, T_I and u denote dimensional quantities. Note that equations (6.47) and (6.49) predict that

$$\text{when} \quad t^* = 0, \quad u = 0 \quad \text{and} \quad T/T_I = X \quad \text{for} \quad 0 \leqslant X < 1. \tag{6.50}$$

The representation (6.46) for A is invalid in a small angle fan neighbouring the ray $\theta = 1$, where

$$1-X-t^* = O(M^{\frac{1}{2}}), \quad = M^{\frac{1}{2}}s \quad \text{say,} \quad \text{and} \quad 0 \leqslant t^* \leqslant \alpha_a^*. \tag{6.51}$$

Across this fan T changes rapidly from T_I to T_a. If we write

$$\theta = 1 - M^{\frac{1}{2}}\bar{\theta} \quad \text{where} \quad \bar{\theta} = s/(1-t^*), \tag{6.52}$$

condition (6.9) implies that in this fan

$$A = \tfrac{1}{4}M[\bar{\theta} + (\bar{\theta}^2+4)^{\frac{1}{2}}]^2. \tag{6.53}$$

When A is given by (6.53), equations (2.42) and (2.43) imply that

$$\frac{T}{T_I} = 1 + \tfrac{1}{2}M^{\frac{1}{2}}\left[\ln\left(\frac{(\bar{\theta}^2+4)^{\frac{1}{2}}+\bar{\theta}+2}{(\bar{\theta}^2+4)^{\frac{1}{2}}+\bar{\theta}-2}\right) - \bar{\theta} - (\bar{\theta}^2+4)^{\frac{1}{2}}\right]. \tag{6.54}$$

In equations (6.51)–(6.54),

$$\bar{\theta}_a \leqslant \bar{\theta} < \infty \tag{6.55}$$

where $\bar{\theta}_a$ corresponds to $T = T_a$. Note that as $\bar{\theta} \to \infty$, equations (6.53) and (6.54) predict that

$$A \to M\bar{\theta}^2 + (1-\theta)^2 \quad \text{while} \quad T/T_I \to 1 - M^{\frac{1}{2}}\bar{\theta} = \theta, \tag{6.56}$$

which are the values (6.46) and (6.47). Also note that

$$\text{when} \quad t^* = 0, \quad \bar{\theta} = (1-X)/M^{\frac{1}{2}}. \tag{6.57}$$

Since condition (6.53) predicts that $A = O(M)$ for finite $\bar{\theta}$, equation (6.8) predicts that

$$\text{when} \quad M = 0, \quad \phi = 1 - t^* = X. \tag{6.58}$$

This condition, together with conditions (2.42), (6.10) and (6.45), then yield the result that the representation (6.49) for u is uniformly valid in the whole interaction region.

(iv) *Rigid–plastic response* $(M \to 0-)$

According to the limits (2.28), the equation of state describing rigid–plastic response contains two material parameters: the limiting traction T_l, and c_l, the limiting value of c. Since $u = c$ in a centred simple wave, c_l is the velocity with which the boundary $X = 1$ moves when it is loaded by the traction T_l.

We calculate the details of the deformation when

$$T_a/T_l = 1 - (-M)^{\frac{1}{2}}\Sigma_a, \quad \text{where} \quad 0 < \Sigma_a = O(1) \quad \text{as} \quad M \to 0. \tag{6.59}$$

It is best to work with the variable

$$\hat{t} = (-M)^{\frac{1}{2}}t, \tag{6.60}$$

rather than with t. This measures time in units of $(2/\pi)(\rho_0 c_l/T_l)D$. On this scale, the front of the incident wave propagates with infinite speed towards $X = 0$ and is represented by the single curve $\hat{t} = 0$ for $0 \leqslant X \leqslant 1$. The front of the reflected wave also propagates with infinite speed until it reaches a layer where

$$1-X = |O(M)|^{\frac{1}{2}}. \tag{6.61}$$

In this layer the reflected wave is strongly refracted by the incident wave and focuses to form a shock at a point $(X_{s_2}, \hat{t}_{s_2})$ which $\rightarrow (1, 1)$ as $M \rightarrow 0$. This shock generates a reflected wave that crosses the layer (6.61) over the period $1 < \hat{t} < \frac{1}{2}\pi$ and then propagates towards $X = 0$ at infinite speed. Thus, as $M \rightarrow 0$ the only part of the interaction region that is not affected by the shock is the rectangular domain (see figure 14)

$$0 \leqslant \hat{t} < \tfrac{1}{2}\pi \quad (0 \leqslant X < 1). \tag{6.62}$$

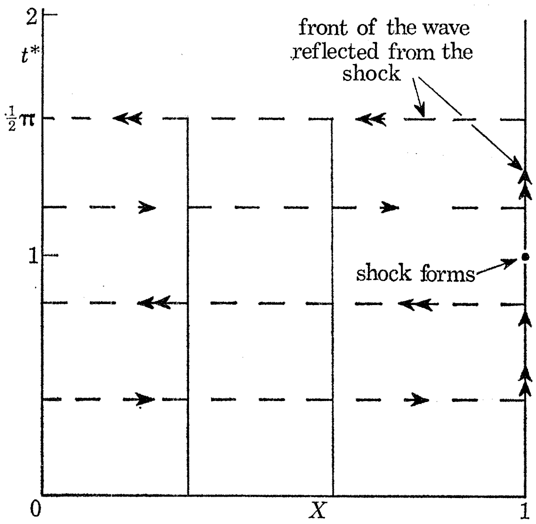


FIGURE 14. The reflexion of a centred wave from a free boundary in a rigid–plastic material $(M \rightarrow 0-)$ when $T_a/T_1 = 1-|O(M)|^{\frac{1}{2}}$. Except in a layer where $1-X = |O(M)|^{\frac{1}{2}}$ the material is in a rigid state. A shock forms at $X = 1$ when $\hat{t} = (-M)^{\frac{1}{2}}\, t = 1$. ——, The trajectories of constant stress; –➤–, β-characteristics; –◀◀–, α-characteristics.

Over most of the region (6.62), outside the layer (6.61), the material is in a rigid state. There, conditions (6.9) and (2.28)–(2.33) imply that

$$A = (1-X)^2, \quad T = T_1 X \quad \text{and} \quad e = -\frac{4M}{3\pi^2}\frac{\rho_0 c_1^2}{T_1}[(1-X)^{-3}-1]. \tag{6.63}$$

In addition, conditions (6.8) and (6.10) imply that

$$u = \frac{2}{\pi} c_1 \hat{t}. \tag{6.64}$$

To prove the result (6.64) we have also used the fact, which follows from equations (2.28) and [I, (8.59)–(8.64)] that

$$\text{as} \quad M \rightarrow 0-, \quad \mu \rightarrow -\frac{\pi^2}{2}\frac{T_1}{\rho_0 c_1^2} \quad \text{and} \quad c_a \rightarrow \frac{c_1}{A_0} = \frac{2}{\pi}\frac{\rho_0 c_1^2}{T_1}(-M)^{\frac{1}{2}}. \tag{6.65}$$

The displacement corresponding to the strain (6.63) and the velocity (6.64) is

$$d = x - X = \frac{2}{\pi^2}\frac{\rho_c c_1^2}{T_1}\{\hat{t}^2 - \tfrac{1}{3}M[(1-X)^{-2} - 2X - 1]\}. \tag{6.66}$$

In the layer (6.61), where the material 'flows',

$$\bar{c} = \frac{\pi}{2}\frac{c}{c_1} = O(1) \quad \text{as} \quad M \to 0. \tag{6.67}$$

Then, conditions (2.28)–(2.30) imply that

$$A = -M\cot^2\bar{c}, \quad T/T_1 = 1-(-M)^{\frac{1}{2}}\cot\bar{c} \tag{6.68}$$

and that
$$\frac{T_1}{\rho_0 c_1^2}e = \left(\frac{2}{\pi}\right)^2(-M)^{-\frac{1}{2}}(\tan\bar{c}-\bar{c}) \quad \text{for} \quad 0<\bar{c}<\tfrac{1}{2}\pi. \tag{6.69}$$

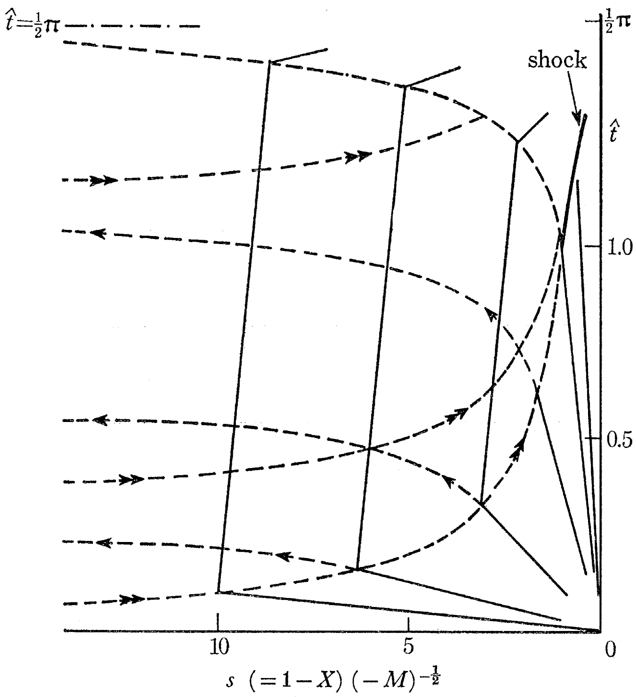


FIGURE 15. The wave pattern produced in a rigid–plastic material in a layer neighbouring the loaded boundary $X = 1$. In this layer the material flows. ——, The trajectories of constant stress; –◀–, β-characteristics; –▶▶–, α-characteristics.

It is best to work with $\hat{t}$ and the distance measure

$$s = (1-X)(-M)^{-\frac{1}{2}}, \tag{6.70}$$

which varies over the range $0 \leqslant s < \infty$. The wave pattern in the $(\hat{t}, s)$ plane is shown in figure 15. The incident fan occupies the region

$$0 \leqslant \hat{t}/s < \Sigma_a^{-2}, \tag{6.71}$$

and, before a shock forms, the front of the reflected wave is represented by the hyperbola

$$\hat{t}s = 1 \quad \text{for} \quad \Sigma_a \leqslant s < \infty \quad \text{and} \quad 0<\hat{t}\leqslant\Sigma_a^{-1}. \tag{6.72}$$

In the incident fan conditions (5.4), (5.6), (6.59), (6.67) and (6.68) imply that

$$\tan\bar{c} = \tan\left(\tfrac{1}{2}\pi\frac{u}{c_1}\right) = \left(\frac{\hat{t}}{s}\right)^{\frac{1}{2}}. \tag{6.73}$$

When the expression (6.73) for $\bar{c}$ in terms of $(\hat{t}, s)$ is inserted in equations (6.68) and (6.69) these determine A, T and e as explicit functions of $(\hat{t}, s)$. The displacement corresponding to this strain field and the velocity field given by equation (6.73) is

$$d = x - X = \left(\frac{2}{\pi}\right)^2 \frac{\rho_0 c_1^2}{T_1} (\bar{c}\operatorname{cosec}^2\bar{c} - \cot\bar{c})\,\hat{t}. \tag{6.74}$$

In the interaction region equations (6.10) and (6.68) imply that

$$\sin 2\bar{c} = 2(\hat{t}+s)^{-1}. \tag{6.75}$$

Equations (6.8), (6.10), (6.68) and (6.75) imply that

$$u = \frac{2}{\pi} c_1 (\bar{c} - s + \cot\bar{c}). \tag{6.76}$$

The displacement field corresponding to the strain and velocity fields given by equations (6.69) and (6.76), with $\bar{c}$ given by condition (6.75), is

$$d = \frac{2}{\pi^2}\frac{\rho_0 c_1^2}{T_1} (\hat{t}^2 + 4\bar{c}\operatorname{cosec}(2\bar{c}) - \tan^2\bar{c} - 2). \tag{6.77}$$

In addition, conditions (4.11), (6.5), (6.60) and (6.70) imply that the trajectories of the α-characteristics as they cross the layer (6.61) are given by

$$(\hat{t} - \hat{\alpha})(s + \hat{\alpha}) = 1, \quad \text{where} \quad \hat{\alpha} = (-M)^{\frac{1}{2}}\alpha. \tag{6.78}$$

Also, conditions (6.11) and (6.12) can be used to show that the trajectories of the β-characteristics are described by the equations

$$\hat{t} = \hat{\beta} + (\tan\bar{c} - 2\bar{c}) \quad \text{and} \quad s = -\hat{\beta} + 2\bar{c} + \cot\bar{c}, \quad \text{where} \quad \hat{\beta} = (-M)^{\frac{1}{2}}\beta. \tag{6.79}$$

Note that the expressions (6.63), (6.64) and (6.66) can be obtained from the expressions (6.68)–(6.70) and (6.75)–(6.77) as $s \to \infty$ at fixed $\hat{t}$. Also, in equations (6.78) and (6.79)

$$(\hat{\alpha}, \hat{\beta}) \to \hat{t} \quad \text{as} \quad s \to \infty \quad (\bar{c} \to 0). \tag{6.80}$$

In equations (6.67)–(6.80) $\bar{c}$ varies over the range

$$0 \leqslant \bar{c} \leqslant \bar{c}_a, \quad \text{where} \quad \cot\bar{c}_a = \Sigma_a. \tag{6.81}$$

If $\Sigma_a < 1$, so that $\bar{c}_a > \frac{1}{4}\pi$, conditions (6.72) and (6.75) predict that a shock forms at the point

$$(\hat{t}, s) = (1, 1) \quad \text{where} \quad \bar{c} = \tfrac{1}{4}\pi. \tag{6.82}$$

This produces a reflected wave whose front is given by equations (6.74) with $\hat{\beta} = \frac{1}{2}\pi$. After traversing the layer (6.61), this front moves with infinite speed towards $X = 0$, where it arrives when $\hat{t} = \frac{1}{2}\pi$. Figure 15 illustrates this situation. The broken curves represent characteristics, the full curves constant levels of stress and strain.

Figure 16 depicts the variations in strain at several typical particles in the layer (6.61). These variations follow from equations (6.69), (6.70), (6.73) and (6.75). In figure 17 we have graphed the displacement of these same particles when $T_a = T_1$ ($\Sigma_a = 0$). At the end of the bar

$$d = \frac{2}{\pi}\frac{\rho_0 c_1^2}{T_1}\hat{t} \quad (s = 0); \tag{6.83}$$

in the rigid region $$d = \frac{2}{\pi^2}\frac{\rho_0 c_1^2}{T_1}\hat{t}^2 \quad (s \to \infty). \tag{6.84}$$

Consequently, by the time $(\hat{t} = 1)$ that the shock forms, the particles in the layer (6.61) occupy a region of width $1.36\rho_0 c_1^2/T_1 D$.

(v) $\gamma = 3$ $(M \to \infty)$

As a last illustration of the general theory described in § 6.1 we consider the limiting behaviour as $M \to \infty$. In this limit equations (6.5), (6.8) and (6.9) imply that

$$\tau = 0, \quad \phi = \alpha = \tfrac{1}{2}t[1 + (1 - 4X/t)^{\frac{1}{2}}] \tag{6.85}$$

and that $$A = \tfrac{1}{4}[1 + (1 - 4X/t)^{\frac{1}{2}}]^2. \tag{6.86}$$

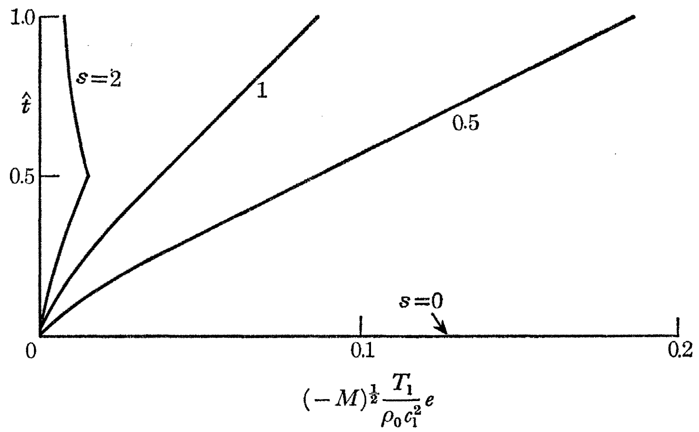


FIGURE 16. The variations in strain at several particles near the loaded boundary of a rigid–plastic material during the reflexion of a centred wave from a free interface.

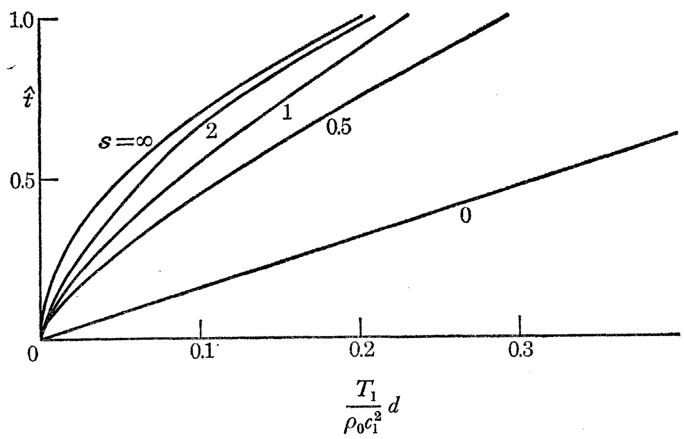


FIGURE 17. The displacements of several particles near the loaded boundary of a rigid–plastic material during the reflexion of a centred wave from a free interface.

The rays θ = constant, on which T, e and c are constant, are centred at $(X, t) = (0, 0)$. These state variables can be calculated as explicit functions of (X, t) from equation (6.86) and the facts that (see I, (8.55)–(8.57))

$$\frac{T}{T_1} = 1 - A^{\frac{3}{2}}, \quad \frac{\rho_0 A_0^2}{T_1}e = 3(A^{-\frac{1}{2}} - 1) \tag{6.87}$$

and
$$\frac{\rho_0 A_0^2}{T_1} c = 3(1 - A^{\frac{1}{2}}). \tag{6.88}$$

It follows from equations (6.10), (6.85), (6.86) and (6.88), together with the fact that

$$\mu = -\frac{2}{3}\frac{\rho_0 A_0^2}{T_1} \tag{6.89}$$

that
$$\frac{\rho_0 A_0^2}{T_1} u = 3[1 - A^{\frac{1}{2}} + \ln(tA^{\frac{1}{2}})]. \tag{6.90}$$

For a gas with $\gamma = 3$,
$$T_1 = -p_0 \quad \text{and} \quad A_0^2 = 3p_0/\rho_0. \tag{6.91}$$

Finally, equations (6.11), (6.86) and (6.88) imply that

$$\beta = tA^{\frac{1}{2}} \exp[2(1 - A^{\frac{1}{2}})],$$

while equations (6.16) and (6.18) imply that

$$A_s = \tfrac{1}{4}, \quad S_2 = \tfrac{7}{8}, \quad X_{s_2} = \tfrac{1}{4} \quad \text{and} \quad t_{s_2} = 2. \tag{6.92}$$

Figure 18 depicts the characteristic curves and the trajectories of constant levels of stress and strain when $T_a/T_1 = \frac{7}{8}$. Then, a shock forms at the reflected front just as it emerges from the interaction region.

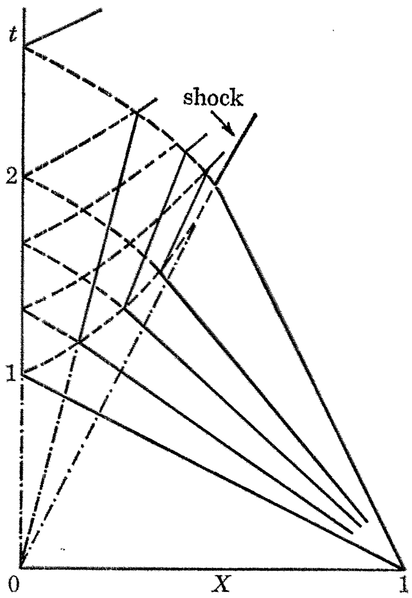


FIGURE 18. The characteristic curves (– –) and the trajectories of constant levels of stress (——) during the reflexion of a centred wave from a free interface when $M = \infty$ ($\gamma = 3$).

7. THE INTERACTION OF THE INCIDENT CENTRED WAVE AND THE WAVE REFLECTED FROM A PERFECTLY RIGID INTERFACE

When a centred wave is incident at a rigid boundary the reflected wave continues to soften the material. Consequently, no shocks form. In fact, for an ideally soft material, if T_a/T_1 is greater than some critical value that only depends on M, the incident wave is so strongly refracted by the reflected wave that part of it never reaches $X = 0$.

The functions F, ϕ and τ are a little more difficult to determine at a rigid interface than at a free interface. The procedure for doing so, though, is quite straightforward. First note that conditions (4.8) imply that ϕ and τ can always be expressed in terms of α and $m(\alpha)$ $(= A^{\frac{1}{2}}$ at $X = 0)$ as

$$\phi = \frac{m}{m^2 - M}(1 - M\alpha) \quad \text{and} \quad \tau = \frac{m^2\alpha - 1}{m^2 - M}. \tag{7.1}$$

The problem then reduces to determining $m(\alpha)$ at a rigid interface. To do this use the fact that

$$\text{at} \quad X = 0, \quad \text{since} \quad u = 0, \quad F = G = \tfrac{1}{2}c, \tag{7.2}$$

so that the equation of state (2.5) implies that

$$\frac{\mathrm{d}F}{\mathrm{d}\alpha} = (\mu + \nu m^2)^{-1}\frac{\mathrm{d}m}{\mathrm{d}\alpha}. \tag{7.3}$$

When the expressions (7.1) and (7.3) are inserted in condition (4.12) this yields the first order equation

$$m(m^2 - M)\frac{\mathrm{d}\alpha}{\mathrm{d}m} + 1 - M\alpha = 0 \tag{7.4}$$

for $m(\alpha)$. When this is integrated, subject to the condition that $m = 1$ when $\alpha = 1$, we obtain

$$m = \left[\frac{M(M-1)}{(M\alpha - 1)^2 + M - 1}\right]^{\frac{1}{2}}, \quad F = \tfrac{1}{2}\hat{c}(m^2) \tag{7.5}$$

$$\phi = \frac{1 - M}{1 - M\alpha}\left[\frac{(M\alpha - 1)^2 + M - 1}{M(M-1)}\right]^{\frac{1}{2}} \quad \text{and} \quad \tau = \frac{1 - \alpha}{1 - M\alpha}. \tag{7.6}$$

When the expressions (7.5) for $\phi(\alpha)$ and $\tau(\alpha)$ are inserted in (4.8) and (4.9) these equations can be solved for A and α. The solutions are

$$A = M(X^2 + M - 1)/[(Mt - 1)^2 + M - 1], \tag{7.7}$$

and

$$\alpha = [(M-1)\,t + (t-1)\,X]/[M - 1 + (Mt - 1)\,X]. \tag{7.8}$$

According to equations (4.13) and (7.5)

$$u = \hat{c}(A) - \hat{c}(m^2), \tag{7.9}$$

where $A(X, t)$ is given by equation (7.7) and m^2 is determined as an explicit function of (X, t) from equations (7.5) and (7.8).

Although the trajectories of the β-characteristics can be calculated from condition (4.14) by using the information in equations (7.5)–(7.9), it is best to proceed directly and use the fact that

$$\text{at constant } \beta, \quad \mathrm{d}X/\mathrm{d}t = -A. \tag{7.10}$$

When the expression (7.7) for A is inserted, equation (7.10) integrates to give

$$\frac{(M-1)\,t + (1-t)\,X}{(M-1) + (1 - Mt)\,X} = \frac{(\bar{\bar{\beta}} - 1)^2 + (1 - M)\,\bar{\bar{\beta}}^2}{M(\bar{\bar{\beta}} - 1)^2 + 1 - M}, \quad = \bar{\bar{\beta}} \quad \text{say}, \tag{7.11}$$

where $\bar{\bar{\beta}}$ denotes the time the β-characteristic crosses the front of the reflected wave. According to equation (6.3),

$$1 \leqslant \bar{\bar{\beta}} \leqslant A_a^{-\frac{1}{2}}. \tag{7.12}$$

The trajectories of the front and the back of the centred wave as it traverses the interaction region are described by (7.11) with $\bar{\bar{\beta}} = 1$ and $\bar{\bar{\beta}} = A_a^{-\frac{1}{2}}$.

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Figures 19 and 20 show the trajectories of the characteristics in region II for a typical non-ideal material ($M = 0.357$) and a typical ideal material ($M = 1.1$). In a non-ideal material the incident wave is completely reflected from the rigid interface in a finite time, in an ideal material the influence of the incident wave *may* persist indefinitely. For, according to equation (7.11), when M lies in the range $(0, 1)$ any β-characteristic that crosses the front of the reflected wave at $t = \bar{\bar{\beta}}$ reaches the boundary $X = 0$ at $t = \bar{\beta}$. This implies that for any finite value of T_a the incident centred wave is completely reflected from $X = 0$ prior to $t = M^{-1}$ – the value of $\bar{\beta}$ corresponding to $A_a = M$ and $\bar{\bar{\beta}} = M^{-\frac{1}{2}}$. For an ideal material, however, only those β-characteristics that cross the front at time $\bar{\bar{\beta}}$ lying in the range

$$1 \leqslant \bar{\bar{\beta}} < 1 + \left(\frac{M-1}{M}\right)^{\frac{1}{2}}, \quad = \bar{\bar{\beta}}_{\text{crit.}} \quad \text{say,} \tag{7.13}$$

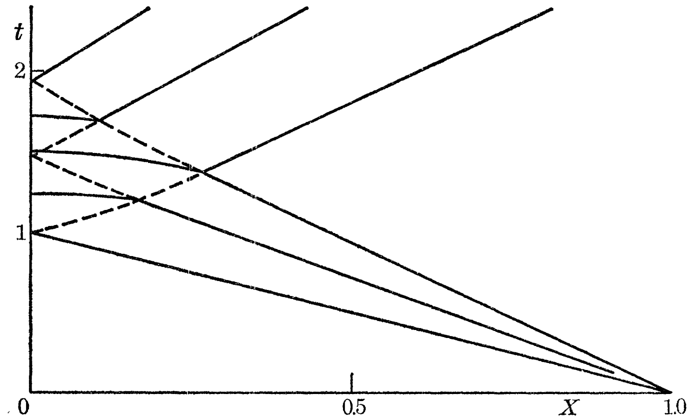

Figure 19. A typical reflexion of a centred wave from a rigid boundary in a non-ideal material ($M = 0.36$). The centred wave is always completely reflected in a finite time. ——, Trajectories of constant levels of stress; – –, the characteristic curves.

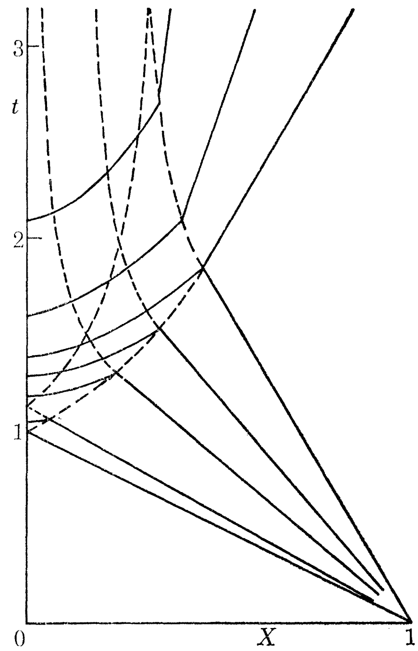

Figure 20. A typical reflexion of a centred wave from a rigid boundary in an ideal material ($M = 1.1$). If the applied traction is sufficiently large the material yields at the rigid boundary and the centred wave is not reflected in a finite time. ——, Trajectories of constant levels of stress; – –, the characteristic curves.

actually reach the boundary $X = 0$. All other characteristics at which $\bar{\bar{\beta}} > \bar{\bar{\beta}}_{\text{crit}}$ are refracted so strongly by the reflected α-wave that they never reach $X = 0$ but, as $t \to \infty$, asymptote to

$$X = (1-M)/(M\bar{\bar{\beta}}-1). \tag{7.14}$$

What happens, of course, is that if T_a/T_1 is large enough to produce characteristics at which $\bar{\bar{\beta}} > \bar{\bar{\beta}}_c$, the reflected wave is sufficiently strong to soften the material to such an extent that it 'yields'. Figure 21 depicts the relation between M and $(T_a/T_1)_{\text{crit}}$, the least value of T_a/T_1 to produce yield. This relation is obtained by using conditions (7.12) and (7.13) which imply that

$$A_{\text{crit}} = \left[1+\left(\frac{M-1}{M}\right)^{\frac{1}{2}}\right]^{-2}. \tag{7.15}$$

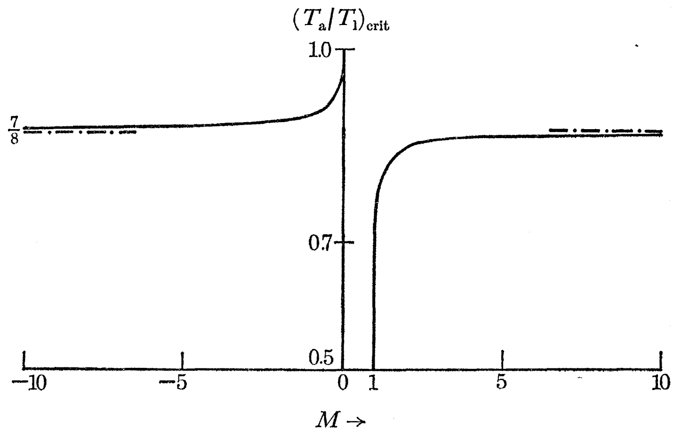

FIGURE 21. The relation between M and the least value of T_a/T_1 that will produce yield at a rigid boundary during the reflexion of a centred wave.

Note that $(T/T_1)_{\text{crit}} \equiv S_2(M)$ – the least value of T/T_1 that causes a shock to form in region II when the interface is perfectly free. An alternative argument is as follows. Since

$$c = \begin{cases} G & \text{in the incident wave,} \\ 2G & \text{at the interface,} \end{cases} \tag{7.16}$$

only that part of the incident wave in which $0 \leqslant G/c_1 < \frac{1}{2}$, where $\hat{A}(c_1) = 0$, actually reaches $X = 0$. In fact,

$$A_{\text{crit}} = \hat{A}(\tfrac{1}{2}c_1). \tag{7.17}$$

If $c_a/c_1 > \frac{1}{2}$, part of the reflected wave may also be trapped. For even though F/c_1 increases monotonically with t in the range $(0, \frac{1}{2})$ at $X = 0$, only that part of the reflected wave in which

$$0 \leqslant F/c_1 < 1 - c_a/c_1$$

actually traverses the interaction region. This situation is illustrated in figure 20.

7.1. *Limiting cases*

(i) *Hookean material* $(M \to 1-)$

As in the case of reflexion from a free boundary, the interaction region is bounded by the curves (6.22) in the $(\bar{t}, \bar{X})$ plane. Now though, according to equations (7.7) and (7.8), the dominant approximations to A and α are

$$A = 1-(1-M)[1+\bar{X}^2-(1-\bar{t})^2] \quad \text{and} \quad \alpha = 1+(1-M)(\bar{t}-\bar{X}). \tag{7.18}$$

It follows from equations (6.20) and (7.9) that

$$c = e = T = -T_I \ln[(1-\bar{t})^2 - \bar{X}^2], \tag{7.19}$$

and that

$$u = T_1 \ln\left[\frac{1-\bar{t}+\bar{X}}{1-\bar{t}-\bar{X}}\right]. \tag{7.20}$$

Since conditions (6.22) imply that the incident wave is completely reflected from $\bar{X} = 0$ when $\bar{t}$ is given by equation (6.26), it follows from equations (7.19) that the traction at the wall after reflexion is $2T_a$ – twice its value in the incident wave.

(ii) *Perfectly elastic–perfectly plastic response* $(M \to 1+)$

According to figure 21, as $M \to 1+$ the traction at the rigid boundary attains the limit value T_l whenever

$$\tfrac{1}{2} < T_a/T_l \leqslant 1. \tag{7.21}$$

Here, to illustrate what happens in this limit we consider the special case when

$$T_a/T_l = \tfrac{1}{2} + |O(M-1)|. \tag{7.22}$$

Then, according to equations (6.31) and (6.3), the incident wave is represented by a small angle fan in which

$$A = \frac{1-X}{t} = 1 + O[(M-1)^{\frac{1}{2}}], \tag{7.23}$$

while the interaction region is confined to a layer where

$$X = O[(M-1)^{\frac{1}{2}}], \quad = (M-1)^{\frac{1}{2}}\bar{\bar{X}} \quad \text{say}. \tag{7.24}$$

This latter result follows from the fact that when condition (7.22) holds, equations (6.28) and (6.29) predict that

$$A_a = 1 - 4(M-1)^{\frac{1}{2}} + O[(M-1)] \tag{7.25}$$

so that, according to conditions (6.3), the back of the centred wave crosses the front of the reflected wave when

$$X = 2(M-1)^{\frac{1}{2}} + O[(M-1)] \quad \text{and} \quad t-1 = 2(M-1)^{\frac{1}{2}} + O[(M-1)]. \tag{7.26}$$

The interaction layer (7.24) is divided into three regions. In the first, $\alpha - 1 = O[(M-1)]$ and $A - 1 = O[(M-1)]$; in the second $\alpha - 1 = O[(M-1)^{\frac{1}{2}}]$ and A varies over the full range $(1, 0)$; in the third $\alpha - 1 = O(1)$ and $A = O[(M-1)]$.

In the first region the dominant approximation to α is given by the second of equations (7.18). (Note that, now, $\bar{X}$ and $\bar{t}$ are $\leqslant 0$.) However, the dominant approximation to A is only given by the first of equations (7.18) when $X = O[(M-1)]$. In this subregion it follows from conditions (6.32) that

$$T = e = c = -T_l[\ln(M-1)]^{-1} \ln[1 - \tfrac{1}{4}(\bar{X}^2 - \bar{t}^2 + 2\bar{t})], \tag{7.27}$$

and from conditions (7.5), (7.9), (7.18) and (7.27) that

$$u = T_l[\ln(M-1)]^{-1} \ln\left[\frac{3 + (\bar{t} - \bar{X} - 1)^2}{3 + (\bar{t} - 1)^2 - \bar{X}^2}\right]. \tag{7.28}$$

When $X = O[(M-1)^{\frac{1}{2}}]$, according to equations (7.7) and (7.24),

$$A = 1 - (M-1)^{\frac{1}{2}} \frac{2\bar{\bar{X}}}{1+\bar{\bar{X}}^2}(1 + \bar{X} - \bar{t}). \tag{7.29}$$

The corresponding variations in T, e and c follow from equations (6.32): they are

$$T = e = c = T_1\left(\tfrac{1}{2} - [\ln(M-1)]^{-1}\ln\left[\frac{1}{2}\frac{\bar{\bar{\bar{X}}}}{1+\bar{\bar{\bar{X}}}^2}(1+\bar{X}-\bar{t})\right]\right). \tag{7.30}$$

It follows from equations (7.9) and (7.18) that

$$u = T_1\left(\tfrac{1}{2} + [\ln(M-1)]^{-1}\ln\left[\frac{1+\bar{\bar{\bar{X}}}^2}{2\bar{\bar{\bar{X}}}}\,\frac{3+(\bar{t}-\bar{X}-1)^2}{1+\bar{X}-\bar{t}}\right]\right). \tag{7.31}$$

In the second region $\quad t-1 = O[(M-1)^{\frac{1}{2}}], \quad = (M-1)^{\frac{1}{2}}\bar{\bar{t}}$ say, (7.32)

and, according to equations (7.7) and (7.8), essentially

$$A = \frac{1+\bar{\bar{X}}^2}{1+\bar{\bar{t}}^2} \quad \text{while} \quad \alpha = 1+(M-1)^{\frac{1}{2}}\frac{\bar{\bar{t}}-\bar{\bar{X}}}{1+\bar{\bar{X}}\bar{\bar{t}}}. \tag{7.33}$$

The variations of T, c and e with $(\bar{\bar{X}}, \bar{\bar{t}})$ then follow from equations (2.25) and (2.26), which state that

$$T = c = T_1\left[1 - [\ln(M-1)]^{-1}\ln\left(\frac{1-A^{\frac{1}{2}}}{1+A^{\frac{1}{2}}}\right)\right] \tag{7.34}$$

while
$$e = T_1\left(1 - [\ln(M-1)]^{-1}\left[2A^{-\frac{1}{2}} + \ln\left(\frac{1-A^{\frac{1}{2}}}{1+A^{\frac{1}{2}}}\right)\right]\right). \tag{7.35}$$

The variation of u with $(\bar{\bar{X}}, \bar{\bar{t}})$ follows from equations (7.9), (7.33) and (7.34): these yield

$$u = T_1[\ln(M-1)]^{-1}\ln\left(\frac{(1+\bar{\bar{\alpha}}^2)^{\frac{1}{2}}-1}{(1+\bar{\bar{\alpha}}^2)^{\frac{1}{2}}+1}\,\frac{1+A^{\frac{1}{2}}}{1-A^{\frac{1}{2}}}\right), \tag{7.36}$$

where $A(\bar{\bar{X}}, \bar{\bar{t}})$ is given by equation (7.33) and

$$\bar{\bar{\alpha}} = \frac{\alpha-1}{(M-1)^{\frac{1}{2}}} = \frac{\bar{\bar{t}}-\bar{\bar{X}}}{1+\bar{\bar{X}}\bar{\bar{t}}}. \tag{7.37}$$

Finally, in the third region equations (7.7) and (7.8) imply that

$$A = (M-1)\frac{1+\bar{\bar{X}}^2}{(t-1)^2} \tag{7.38}$$

and that
$$\alpha = 1+(M-1)^{\frac{1}{2}}\frac{t-1}{(M-1)^{\frac{1}{2}}+(t-1)\,\bar{\bar{X}}}. \tag{7.39}$$

It then follows from equations (2.25) and (2.26) that

$$T = c = T_1\left[1 + 2\frac{(M-1)}{\ln(M-1)}\,\frac{(1+\bar{\bar{X}}^2)^{\frac{1}{2}}}{t-1}\right] \tag{7.40}$$

and that
$$e = -2T_1[(M-1)^{\frac{1}{2}}\ln(M-1)]^{-1}\frac{t-1}{(1+\bar{\bar{X}}^2)^{\frac{1}{2}}}, \tag{7.41}$$

while (7.9), (7.39) and (7.40) imply that

$$u = -2T_1[\ln(M-1)]^{-1}\{\bar{\bar{X}} + (M-1)^{\frac{1}{2}}[1-(1+\bar{\bar{X}}^2)^{\frac{1}{2}}]/(t-1)\}. \tag{7.42}$$

Note that even though equation (7.41) predicts that e is unbounded like $[(M-1)^{\frac{1}{2}}\ln(M-1)]^{-1}$, since the thickness of the interaction region is only $O[(M-1)^{\frac{1}{2}}]$ the material does not undergo a finite displacement even though $T \simeq T_1$.

(iii) *Rigid–elastic response* $(M \to 0+)$

We consider the situation when T_a satisfies condition (6.39). Then, as in the case of reflexion from a perfectly free interface, it is best to use $t^* = Mt$ as the time measure. On this scale the fronts of the incident and reflected waves are represented by $t^* = 0$ and, no matter how large T_a/T_I, the incident wave is always completely reflected from the rigid interface before $t^* = 1$.

According to equations (7.7), (7.8) and (7.11), except when $t^* = O(M)$ $(t = O(1))$ the material is in the elastic state with

$$\bar{A} = M^{-1}A = \frac{1-X^2}{1-(1-t^*)^2}, \quad \alpha^* = M\alpha = \frac{t^*(1-X)}{1+(1-t^*)X} \tag{7.43}$$

and
$$\bar{\beta}^* = M\bar{\beta} = \frac{t^*(1+X)}{1-X+t^*X}. \tag{7.44}$$

In these equations
$$0 < (\alpha^*, \bar{\beta}^*) \leqslant \alpha^*_a, \tag{7.45}$$

where
$$\alpha^*_a = 2(1+\bar{A}_a)^{-1}, \tag{7.46}$$

which is $\leqslant 1$ because $1 \leqslant \bar{A}_a < \infty$. Also, by condition (6.3),

$$0 \leqslant X < 1-|O(M)|^{\frac{1}{2}}. \tag{7.47}$$

To determine $\bar{A}_a$ in terms of T_a/T_I use the fact, which follows from conditions (2.42) and (2.43), that

$$\frac{T}{T_I} = 1+\tfrac{1}{2}M^{\frac{1}{2}}\left[\ln\left(\frac{\bar{A}^{\frac{1}{2}}+1}{\bar{A}^{\frac{1}{2}}-1}\right)-2\bar{A}^{\frac{1}{2}}\right]. \tag{7.48}$$

This relation, with $\bar{A}$ given by the first of equations (2.43), determines T as an explicit function of (X, t^*). $c(X, t^*)$ and $e(X, t^*)$ can likewise be determined from the facts that

$$\frac{\rho_0 A_\infty}{T_I}c = \tfrac{1}{2}M^{\frac{1}{2}}\ln\left(\frac{\bar{A}^{\frac{1}{2}}+1}{\bar{A}^{\frac{1}{2}}-1}\right), \tag{7.49}$$

and
$$\frac{\rho_0 A^2_\infty}{T_I}e = \frac{T}{T_I}-1. \tag{7.50}$$

These relations follow from equations (2.42)–(2.44). Conditions (7.9), (7.43) and (7.49) imply that

$$\frac{\rho_0 A_\infty}{T_I}u = \tfrac{1}{2}M^{\frac{1}{2}}\ln\left(\frac{\bar{A}^{\frac{1}{2}}+1}{\bar{A}^{\frac{1}{2}}-1}\frac{\bar{m}-1}{\bar{m}+1}\right), \tag{7.51}$$

where
$$\bar{m} = [1-(1-\alpha^*)^2]^{-\frac{1}{2}} = [1+X(1-t^*)][(1-X^2)(1-(1-t^*)^2)]^{-\frac{1}{2}}. \tag{7.52}$$

Figure 22 depicts the characteristic net and the trajectories of constant T and e when

$$T_a/T_I - 1 > |O(1)|.$$

Then $\alpha^*_a = 1$ and the interaction region is the triangular domain bounded by the rays

$$t^* = 0, \quad X = 0 \quad \text{and} \quad t^* = 1-X. \tag{7.53}$$

Note that the β-characteristics and the curves of constant T and e are all centred on the point $(X, t) = (1, 0)$.

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The representations (7.43)–(7.52) are invalid when $t^* = O(M)$. Over this period the material is in a near rigid state. Equations (7.7) and (7.8) imply that

$$A = \frac{1-X^2}{2t-1} \quad \text{and that} \quad \alpha = \frac{t(1-X)+X}{1+X}. \tag{7.54}$$

Conditions (2.45), (2.46) and (7.54) imply that

$$\frac{\rho_0 A_\infty}{T_I} c = M\left[\left(\frac{2t-1}{1-X^2}\right)-1\right], \tag{7.55}$$

while

$$T = T_I\left[1-\frac{1-X^2}{2t-1}\right] \quad \text{and} \quad \frac{\rho_0 A_\infty^2}{T_I} e = \tfrac{1}{3}M^2\left[\left(\frac{2t-1}{1-X^2}\right)-1\right]. \tag{7.56}$$

$u(X, t)$ then follows from equations (7.9), (7.54) and (7.55), it is given by

$$\frac{\rho_0 A_\infty}{T_I} u = M\left(\frac{2t-1}{1-X^2}\right)^{\frac{1}{2}} X. \tag{7.57}$$

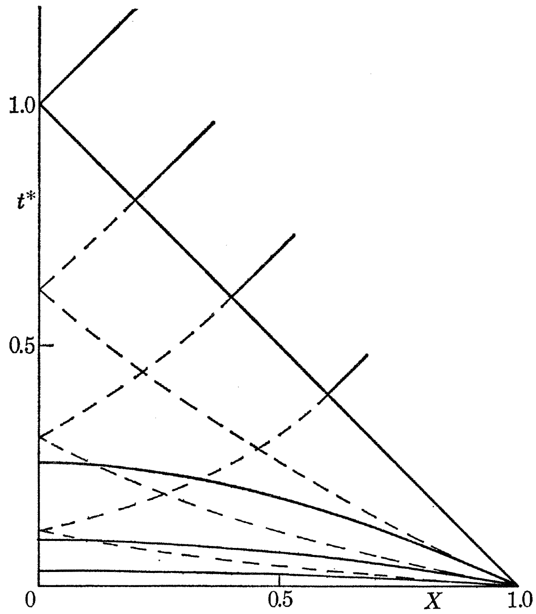

FIGURE 22. The reflexion of a centred wave at a rigid interface in a rigid–elastic material when $T_a/T_I > 1+|O(M)|^{\frac{1}{2}}$. Except when $t^* = O(M)$ the material is in the elastic state. ——, The trajectories of constant stress; – –, the characteristic curves.

(iv) *Rigid–plastic response* $(M \to 0-)$

We consider the response of a rigid–plastic material when T_a is close to T_I in the sense (6.59). Then, the material is in a near plastic state when

$$t^* = -Mt = O(1). \tag{7.58}$$

In this state, according to equations (7.7), (7.8) and (7.11)

$$A = -M\frac{1-X^2}{(1+t^*)^2-1}, \quad \alpha^* = -M\alpha = t^*\frac{1-X}{1+X}, \tag{7.59}$$

and

$$\bar{\beta}^* = -M\bar{\beta} = t^*\frac{1+X}{1-X}, \tag{7.60}$$

T and e can be determined from equations (6.68) and (6.69) with

$$\tan \bar{c} = \left[\frac{(1+t^*)^2-1}{1-X^2}\right]^{\frac{1}{2}}. \tag{7.61}$$

From equations (6.67), (7.9) and (7.59) it follows that

$$u = \frac{2}{\pi} c_1(\bar{c}-\bar{c}_0), \tag{7.62}$$

where $$\tan \bar{c}_0 = [(1+\alpha^*)^2-1]^{\frac{1}{2}}. \tag{7.63}$$

The displacement field corresponding to the velocity field (7.62) is given by

$$d = x - X = \left(\frac{2}{\pi}\right)^2 \frac{\rho_0 c_1^2}{T_1}(-M)^{-\frac{1}{2}}\left[(1+t^*)\tan^{-1}\left(\frac{X}{1+t^*}\tan\bar{c}\right)-\bar{c}X\right], \tag{7.64}$$

where $\bar{c}(X, t^*)$ is determined from condition (7.61). In the (t^*, X) plane the interaction region is bounded by the curves

$$t^* = 0, \quad X = 0 \quad \text{and} \quad t^*\frac{1+X}{1-X} = 2\Sigma_a^{-2}. \tag{7.65}$$

Figure 23 depicts the characteristic curves and the trajectories of constant levels of T and e in the limit as $\Sigma_a \to 0$. Then, at all values of t^*, the interaction region fills the whole slab.

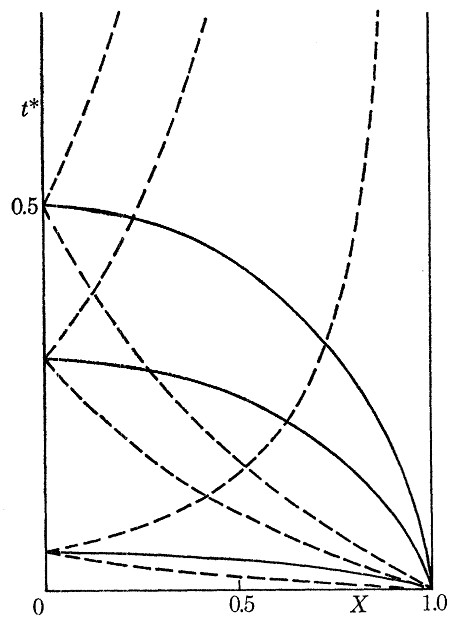

FIGURE 23. The reflexion of a centred wave at a rigid interface in a rigid–plastic material when $T_a/T_1 = 1-|O(M)|^{\frac{1}{2}}$. When $t^* = O(1)$ the material is in a near plastic state. ——, The trajectories of constant stress; – –, the characteristic curves.

The representations (7.59)–(7.65) are invalid when $t^* = O(-M)$, $(t = O(1))$. Over this period the material is in a near rigid state with

$$A = \frac{1-X^2}{2t-1}, \quad \alpha = \frac{t(1-X)+X}{1+X} \quad \text{and} \quad \beta = \frac{t(1+X)-X}{1-X}. \tag{7.66}$$

c, A, T and e can be calculated from equations (2.45) and (2.46) with

$$\eta = \left(\frac{2t-1}{1-X^2}\right)^{\frac{1}{2}} - 1. \tag{7.67}$$

Also, from equations (2.45), (2.9), (7.66) and (7.67) it follows that

$$\frac{\rho_0 A_\infty}{T_I} u = M\left(\frac{2t-1}{1-X^2}\right)^{\frac{1}{2}} X. \tag{7.68}$$

The part of the interaction region where (7.66)–(7.68) are valid is bounded by the curves

$$X = 0, \quad \text{and} \quad t(1-X) = 1 \quad \text{for} \quad 1 \leqslant t < \infty. \tag{7.69}$$

(v) $\gamma = 3$ ($M \to \infty$)

In this limit the representation of the deformation in the interaction region simplifies considerably. For, according to equations (7.7), (7.8) and (7.11)

$$A = t^{-2}, \quad \alpha = \frac{t}{1+tX} \quad \text{and} \quad \bar{\beta} = \frac{t}{1-tX}. \tag{7.70}$$

It follows from equations (6.87)–(6.88) that

$$\frac{T}{T_\mathrm{I}} = 1 - t^{-3}, \quad \frac{\rho_0 A_0^2}{T_\mathrm{I}} e = 3(t-1) \quad \text{and} \quad \frac{\rho_0 A_0^2}{T_\mathrm{I}} c = 3\left(\frac{t-1}{t}\right), \tag{7.71}$$

while by equations (7.9)

$$\frac{\rho_0 A_0^2}{T_\mathrm{I}} u = 3X. \tag{7.72}$$

The front of the reflected wave is

$$t(1-X) = 1 \quad \text{for} \quad 1 \leqslant t \leqslant \left[1 - \frac{T_\mathrm{a}}{T_\mathrm{I}}\right]^{-\frac{1}{3}}, \tag{7.73}$$

the back of the incident wave is given by the last of equations (7.70) with

$$\bar{\beta} = \left[2\left(1 - \frac{T_\mathrm{a}}{T_\mathrm{I}}\right)^{\frac{1}{3}} - 1\right]^{-1}. \tag{7.74}$$

The limit stress is attained at the rigid interface iı

$$T_\mathrm{a}/T_\mathrm{I} \geqslant \tfrac{7}{8}. \tag{7.75}$$

8. The interaction of the incident centred wave and the wave reflected from an interface with a Hookean material

In many situations the interface $X = 0$ separates two elastic materials which have the property that although the response of the material to the right is grossly nonlinear for the stress level that occurs, the response of that to the left remains essentially linear. If T_L, the traction in the linear Hookean material, is measured in units of $\rho_0 A_0^2$ and if c_L is measured in units of A_0, the equation of state of the Hookean material is

$$T_\mathrm{L} = i_0 c_\mathrm{L}, \tag{8.1}$$

where

$$i_0 = \frac{\rho_{\mathrm{L}0} A_{\mathrm{L}0}}{\rho_0 A_0} \quad \text{(in dimensional variables)} \tag{8.2}$$

is the impedance of the interface when $T = 0$. When condition (8.1) holds, the interface condition (3.1) states that

$$\text{at} \quad X = 0, \quad u = i_0^{-1} T(c) \tag{8.3}$$

so that, by equations (2.2) and (2.4),

$$G = \tfrac{1}{2}[c+i_0^{-1}T(c)] \quad \text{and} \quad F = \tfrac{1}{2}[c-i_0^{-1}T(c)]. \tag{8.4}$$

The last of these equations, together with conditions (2.3) and (2.5), imply that

$$\nu\frac{\mathrm{d}F}{\mathrm{d}\alpha} = \left(\frac{1-i_0^{-1}m^2}{m^2-M}\right)\frac{\mathrm{d}m}{\mathrm{d}\alpha}. \tag{8.5}$$

When the expressions (7.1) and (8.5) are substituted in condition (4.12), this yields the first order equation

$$m(m^2-M)\frac{\mathrm{d}\alpha}{\mathrm{d}m}+(1-M\alpha)\,(1+i_0^{-1}m^2) = 0 \tag{8.6}$$

for $\alpha = \hat{\alpha}(m)$. This integrates, subject to the condition that $m = 1$ when $\alpha = 1$, to give

$$\alpha = 1+\frac{1-M}{M}\left[1-m^{-1}\left(\frac{m^2-M}{1-M}\right)^{\frac{1}{2}(1+i_0^{-1}M)}\right]. \tag{8.7}$$

When the expression (8.7) for α is inserted, equations (7.1) determine $\phi = \phi(m)$ and $\tau = \tau(m)$ as explicit functions of the characteristic parameter m. $F = F(m)$ is determined from equation (8.4) with $c = \hat{c}(m^2)$. The statement (7.5) for $m(\alpha)$ at a rigid interface is obtained from equation (8.7) as $i_0 \to \infty$. The variation of u with α at a free interface, which follows from equations (6.5) and (6.10), is obtained from equations (8.3) and (8.6) in the limit as $(i_0, T, c, A-1) \to 0$ but

$$i_0^{-1}\frac{m^2-1}{1-M} \to -\frac{\mu}{M}i_0^{-1}c \to -\frac{\mu}{M}i_0^{-1}T = -\frac{\mu}{M}u. \tag{8.8}$$

8.1. *Yield at the interface*

In general the incident wave will be completely reflected from $X = 0$ when c attains a value c_{aa} which is related to c_{a} by the equation

$$2c_{\mathrm{a}} = c_{\mathrm{aa}}+i_0^{-1}T(c_{\mathrm{aa}}). \tag{8.9}$$

This result follows from the first of equations (8.4) and the fact that $G = c_{\mathrm{a}}$ at the back of the incident wave. Equation (8.9), with $c_{\mathrm{aa}} = c_{\mathrm{l}}$ and $T = T_{\mathrm{l}}$, implies that the least value of $c_{\mathrm{a}}/c_{\mathrm{l}}$ that will cause an ideal material to yield at the interface is

$$\frac{c_{\mathrm{a}}}{c_{\mathrm{l}}} = \frac{1}{2}\left[1+i_0^{-1}\frac{T_{\mathrm{l}}}{c_{\mathrm{l}}}\right]. \tag{8.10}$$

Consequently, since $c_{\mathrm{a}}/c_{\mathrm{l}} \leqslant 1$, a *sufficient* condition that the material shall not yield (no matter how close T_{a} is to T_{l}) is that

$$i_0 < T_{\mathrm{l}}/c_{\mathrm{l}}, \quad = I_0(M), \quad (\leqslant 1). \tag{8.11}$$

The function $I_0(M)$ can easily be calculated by using I, (8.49) and (8.59): it is graphed in figure 24. In this same figure we have also depicted the variations of the critical value of $T_{\mathrm{a}}/T_{\mathrm{l}}$ with M at several values of i_0. For values of $T_{\mathrm{a}}/T_{\mathrm{l}}$ greater than this critical value the material will yield at an interface characterized by the parameter i_0.

8.2. *Shock formation*

A study of equations (8.6) and (8.9) shows that

$$\text{at} \quad X = 0, \quad \mathrm{d}A/\mathrm{d}t = \mathrm{d}m^2/\mathrm{d}^2\alpha < 0 \quad \text{for} \quad -\infty \leqslant M \leqslant \infty \quad \text{and} \quad 0 \leqslant i_0 < \infty. \tag{8.12}$$

Consequently, in general a material always softens at an interface with a Hookean material

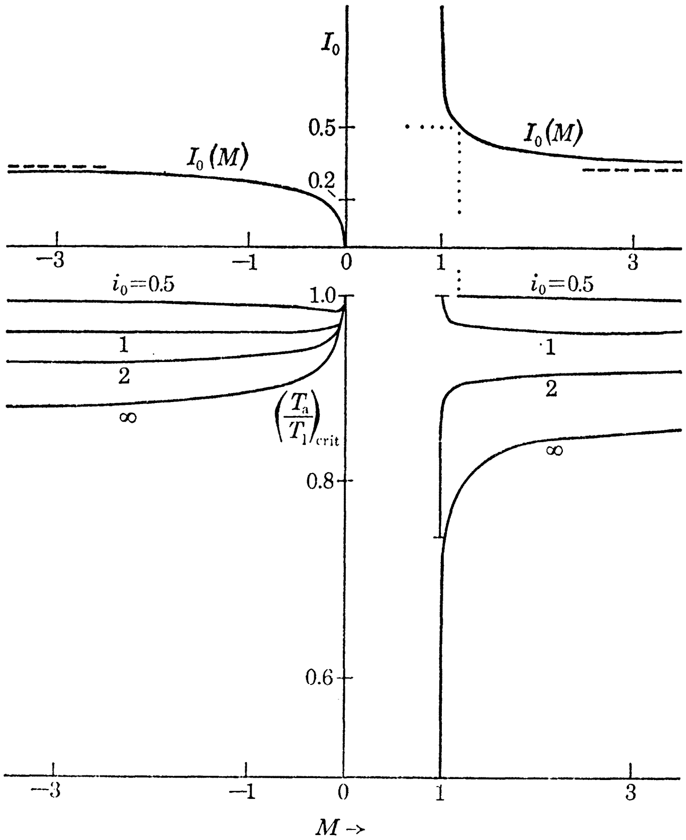

FIGURE 24. The variations with M of the least values of T_a/T_1 that will produce yield at an interface with a Hookean material during the reflexion of a centred wave. The curves are characterized by i_0, the impedance of the interface in the reference configuration. Also shown in this figure is the variation of I_0 with M: if $i_0 < I_0(M)$ the material can never yield no matter how close T_a is to T_1. Note that if $i_0 < \frac{1}{3}$ yield cannot occur for any material.

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during the reflexion of a centred wave. This causes the reflected wave to defocus at the interface. The one exception is when the interface is perfectly free ($i_0 = 0$): then $\mathrm{d}A/\mathrm{d}t \equiv 0$ at $X = 0$. However, when the material is ideally soft (M lies outside the range $(0, 1)$) and when $i_0 \leqslant 1$ the material may begin to harden away from the interface and the reflected wave may focus to form a shock. Hardening and shock formation always occur first at the front of the reflected wave.

Figure 25 depicts the relations between M and the least value of T_a/T_1 that causes the material to harden. The curves are drawn for several fixed values of i_0 lying in the interval $[0, 1]$. To obtain this information we have used the fact that conditions (4.8), with ϕ and τ determined as functions of m by the procedure outlined in section 8, predict that

$$\mathrm{d}A/\mathrm{d}t = 0 \quad \text{when} \quad A^{\frac{1}{2}} = M(m^2 - i_0)/m(M - i_0). \tag{8.13}$$

When this statement for $A^{\frac{1}{2}}$ is inserted in equations (4.8) these yield the equations

$$Mt = 1 + m\frac{M - m^2}{m^2 - i_0}\phi(m) \quad \text{and} \quad X = i_0\frac{M - m^2}{m(M - i_0)}\phi(m) \tag{8.14}$$

for the curve in the (t, X) plane that separates the region where the material is softening from the region where it is hardening. A careful study of the curves (8.14) indicates that a *necessary* condition that they lie in the interaction region is that $i_0 < 1$ and that M lies outside the range $(0, 1)$.

Moreover, since (8.14) states that t increases as m decreases when i_0 and M vary over this range, if the material hardens it does so first at the front of the reflected wave, where

$$\phi = m = 1, \quad t = \frac{M-i_0}{M(1-i_0)} \quad \text{and} \quad X = 1 - M\frac{1-i_0}{M-i_0}. \tag{8.15}$$

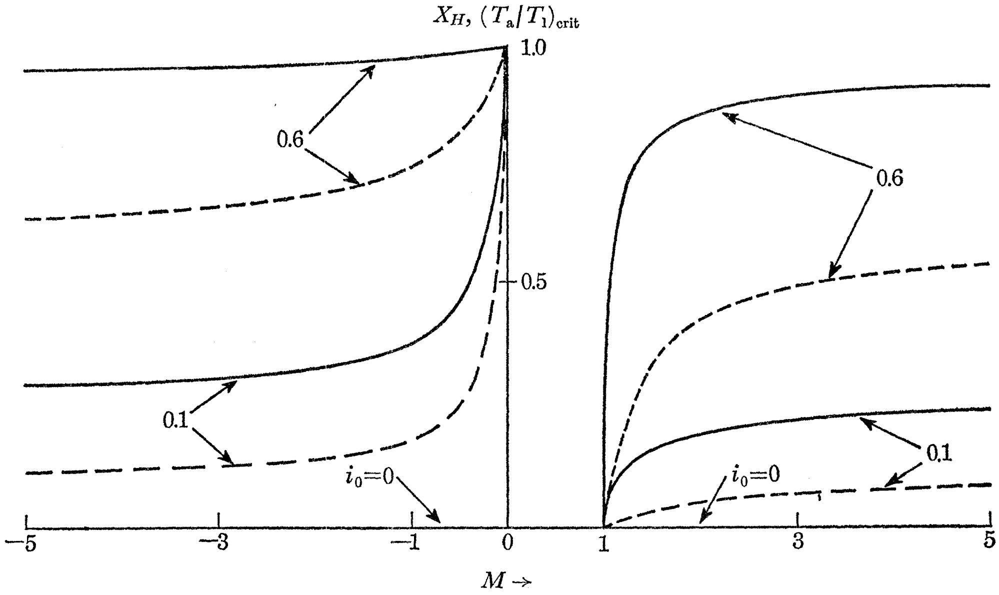

FIGURE 25. The relation between M and the least value of T_a/T_1 that will cause the material to harden during the reflexion of a centred wave from an interface with a Hookean material. The curves are characterized by i_0, the impedance of the interface in the reference configuration. Hardening always first occurs at the front of the reflected waves at $X = X_H(M; i_0)$. ——, $(T_a/T_1)_{crit}$; – –, X_H.

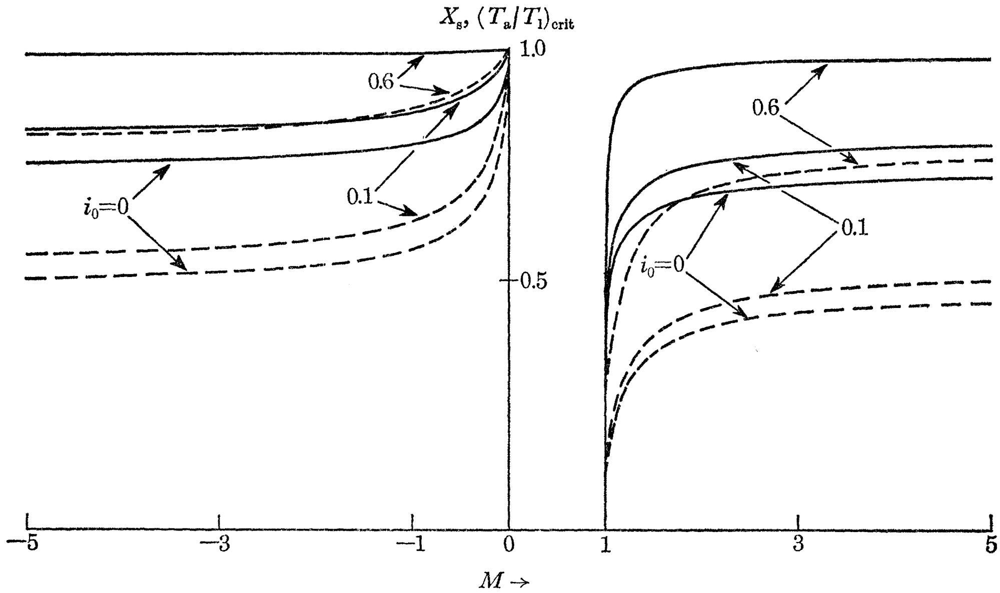

FIGURE 26. The relation between M and the least value of T_a/T_1 that will cause a shock to form during the reflexion of a centred wave from an interface with a Hookean material. The curves are parameterized by i_0, the impedance of the interface in the reference configuration. The shock always forms at the front of the reflected wave at $X = X_s(M: i_0)$. ——, $(T_a/T_1)_{crit}$; – –, X_s.

According to equations (8.13) the corresponding value of

$$A^{\frac{1}{2}} = M\frac{1-i_0}{M-i_0}. \tag{8.16}$$

The information contained in figure 25 now follows from the fact that hardening will indeed occur if T_a/T_1 is large enough for

$$A_a^{\frac{1}{2}} \leqslant M\frac{1-i_0}{M-i_0}. \tag{8.17}$$

Even when condition (8.17) is satisfied the reflected wave need not always focus rapidly enough for a shock to form in the interaction region. After a lengthy argument (which will not be given here) it can be shown that for this to happen

$$A_a \leqslant A_s = \left(\frac{M-i_0}{1-i_0}\right)^2 \left\{1-\left[1-\left(\frac{1-i_0}{M-i_0}\right)^2 M\right]^{\frac{1}{2}}\right\}^2. \tag{8.18}$$

The values of T_a/T_1 corresponding to $A_a = A_s$ are shown in figure 26. A shock forms at the front of the reflected wave when T_a/T_1 is equal to, or greater than, this critical wave.

8.3. $M = \infty$

As a simple illustration of the general results established in this section we consider the special case when $M = \infty$. Then, condition (8.7) implies that

$$\alpha = m^{-1}\exp\left[\tfrac{1}{2}i_0^{-1}(1-m^2)\right], \tag{8.19}$$

while conditions (7.1), (4.8) and (4.9) imply that

$$t = \phi(m)\,A^{-\frac{1}{2}} \quad \text{and} \quad X = \phi(m)\,(m-A^{\frac{1}{2}}), \tag{8.20}$$

where

$$\phi = \alpha m = \exp\left[\tfrac{1}{2}i_0^{-1}(1-m^2)\right]. \tag{8.21}$$

When m is eliminated, equations (8.20) yield the implicit equation

$$m = A^{\frac{1}{2}} + X/A^{\frac{1}{2}}t = \left[1-2i_0\ln(tA^{\frac{1}{2}})\right]^{\frac{1}{2}} \tag{8.22}$$

for the determination of $A(X, t)$ and $m(X, t)$. To determine u use equations (4.13), (6.87), (6.88) and (8.4): these yield

$$u = T_1\left[3(m-A^{\frac{1}{2}}) + i_0^{-1}(1-m^3)\right]. \tag{8.23}$$

Equation (6.88) implies that $T_1/c_1 = \frac{1}{3}$. Consequently, condition (8.11) states that when $M = \infty$ the material can never yield when $i_0 < \frac{1}{3}$. In fact, according to equations (8.10), (6.87) and (6.88) the material will only yield at $X = 0$ if

$$T_a/T_1 \geqslant 1 - \tfrac{1}{8}(1-\tfrac{1}{3}i_0^{-1})^3 \quad \text{for} \quad \tfrac{1}{3} \leqslant i_0 \leqslant \infty. \tag{8.24}$$

According to equations (6.87), (6.88), (8.15) and (8.16) the material will first start to harden at $X = i_0$ if

$$T_a/T_1 > 1-(1-i_0)^3. \tag{8.25}$$

Once the material starts to harden it will continue to do so along the curve represented by the relations

$$t = \frac{m}{m^2-i_0}\phi(m) \quad \text{and} \quad X = i_0 m^{-1}\phi(m), \tag{8.26}$$

where $\phi(m)$ is given by equation (8.21). However, the material will only harden rapidly enough for a shock to form if

$$T_a/T_l > 1-\tfrac{1}{8}(1-i_0)^3. \tag{8.27}$$

When condition (8.27) is satisfied, the shock forms at

$$X = \tfrac{1}{2}(1+i_0) \quad \text{and} \quad t = 2(1-i_0)^{-1}. \tag{8.28}$$

Figure 27 illustrates conditions in the interaction region when $i_0 = 0.1$ and when T_a/T_l is large enough for a shock to form. The curve along which the material begins to harden is clearly shown. Note that once it forms at the front of the reflected wave it moves towards the interface $X = 0$.

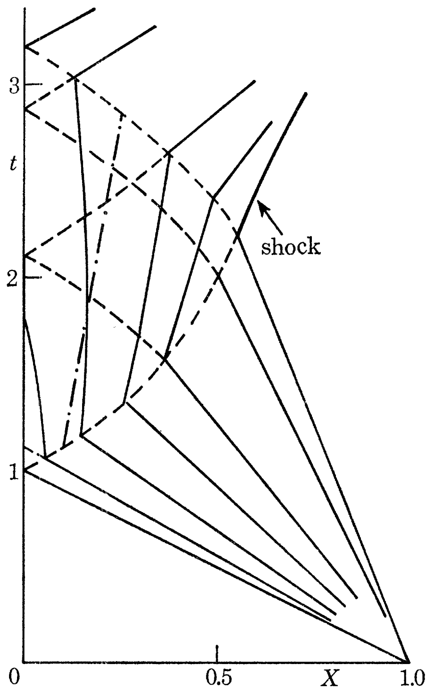

FIGURE 27. The reflexion of a centred wave from an interface with a Hookean material when $M = \infty$ and $i_0 = 0.1$. The reflected wave at first defocuses and then begins to focus. —·—, The curve separating the region where the material softens from the region where the material hardens; ——, the trajectories of constant levels of stress; – –, the characteristic curves.

9. The interaction of the incident centred wave and the wave reflected from an interface with any elastic material

When the response of the material to the left of the interface is nonlinear the functions $\phi(\alpha)$, $\tau(\alpha)$ and $F(\alpha)$ in equations (4.8), (4.9) and (4.13) cannot usually be determined by analytic means. To compute them it is best to introduce the function

$$\bar{G}(\alpha) = G \quad \text{at} \quad X = 0. \tag{9.1}$$

In terms of $\bar{G}$, equations (2.10) and (3.2) imply that

$$F(\alpha) = L(\bar{G}) \quad \text{and} \quad m(\alpha) = \hat{A}^{\frac{1}{2}}[\bar{G}+L(\bar{G})]. \tag{9.2}$$

Equations (7.1), which are valid at any interface, with $m(\alpha)$ given by (9.2) then determine ϕ and τ as functions of α and $\bar{G}$. All that remains is to determine $\bar{G}(\alpha)$. To do this insert the expressions (9.2) for F and m and the expressions (7.1) for ϕ and τ in the compatibility equation (4.12). This yields the equation

$$M\hat{A}^{\frac{1}{2}}[\bar{G}+L(\bar{G})]\,\mathrm{d}\alpha/\mathrm{d}\bar{G}+\mu(M\alpha-1) = 0 \tag{9.3}$$

for $\bar{G}(\alpha)$. This equation integrates, subject to the condition that $\bar{G} = 0$ when $\alpha = 1$, to give

$$\alpha = M^{-1}+(1-M^{-1})\exp\left(-\mu\int_0^{\bar{G}}\frac{\mathrm{d}s}{\hat{A}^{\frac{1}{2}}[s+L(s)]}\right). \tag{9.4}$$

Since $\bar{G} = c_a$ at the back of the incident centred wave, if the material does not yield at the interface $\bar{G}$ varies over the interval $(0, c_a)$ in equation (9.4). However, if the material yields $\bar{G}$ varies over the range $(0, \bar{G}_{aa})$, where

$$\bar{G}_{aa}+L(\bar{G}_{aa}) = c_1. \tag{9.5}$$

Once $\bar{G}(\alpha)$ has been determined from condition (9.4) $F(\alpha)$ and $m(\alpha)$ follow from equations (9.2) while $\phi(\alpha)$ and $\tau(\alpha)$ follow from equations (7.1). The variations of A and u in the interaction region can then be computed from equations (4.8), (4.9) and (4.13).

9.1. *The reflected wave*

When no shocks form, after traversing the interaction region the reflected wave is a simple wave in which $G \equiv c_a$. To determine the signal carried by this wave first note that condition (4.14) implies that

$$c = c_a+F(\alpha) \tag{9.6}$$

at the back of the incident centred wave. Then note that equations (4.8), with

$$A = \hat{A}[c_a+F(\alpha)], \quad = A_B(\alpha) \quad \text{say}, \tag{9.7}$$

yield relations of the form

$$X = X_B(\alpha) \quad \text{and} \quad t = t_B(\alpha) \tag{9.8}$$

for the trajectory of the back of the incident centred wave. The curve (9.8) separates the interaction region from the simple wave region. Since $A \equiv A_B(\alpha)$ in the simple wave region, $\alpha(X, t)$ can be computed from the condition that

$$X-X_B(\alpha) = A_B(\alpha)\,[t-t_B(\alpha)]. \tag{9.9}$$

If the reflected wave does not focus and form a shock as it traverses the simple wave region its front reaches $X = 1$ when $t = 2A_a^{-\frac{1}{2}}$. This follows from the fact that the front emerges from the interaction region when $X = 1-A_a^{\frac{1}{2}}$ and $t = A_a^{-\frac{1}{2}}$ (see equations (6.3)) and then moves with constant speed A_a towards $X = 1$.

Once $\alpha(X, t)$ and $A(X, t)$ have been determined from equations (9.7) and (9.9) $u(X, t)$ can be computed from either of the equations

$$u = c_a-F(\alpha) = 2c_a-\hat{c}(A). \tag{9.10}$$

(i) *Perfectly free interface*

When the interface is perfectly free equations (6.13) express X and t as explicit functions of A at the boundary separating the interaction region and the simple wave region. From these equations it follows that in the simple wave region

$$X = At-Y(A) \quad \text{for} \quad A_a \leqslant A \leqslant 1, \tag{9.11}$$

where

$$Y(A) = M^{-1}[A+(2MA^{\frac{1}{2}}-A-M)\exp(\mu[\hat{c}(A)-c_a])]. \tag{9.12}$$

When $A(X,t)$ has been determined from condition (9.11), $u(X,t)$ can be computed from the second of equations (9.10).

Since the simple wave relation (9.11) predicts that

$$\mathrm{D}A/\mathrm{D}t = A(Y'(A)-t)^{-1}, \tag{9.13}$$

the reflected simple wave will focus and form a shock at the least value of t at which $Y'(A)-t=0$. When the expression (9.12) for $Y(A)$ is examined it turns out that $Y'(A)$ is an increasing function of A when M lies outside the range (0, 1) and a decreasing function when M lies inside this range. Consequently, if a shock forms it does so at the front of the wave, where $A = A_a$, when the material is ideal and at the back of the wave, where $A=1$, when the material is non-ideal. A careful study shows that this later possibility does not in fact occur. Consequently, no shocks form in region III when the material is non-ideal.

By contrast, a shock must *always* form in an ideal material before the front of the wave reflected from $X=0$ reaches $X=1$. For, conditions (9.11) and (9.13) predict that $\mathrm{D}A/\mathrm{D}t$ is unbounded at the point (X_{s_3}, t_{s_3}), where

$$X_{s_3} = 1-2A_a^{\frac{1}{2}}M\left(\frac{1-A_a^{\frac{1}{2}}}{M-A_a}\right) \quad \text{and} \quad t_{s_3} = 2\left(\frac{M-A_a^{\frac{1}{2}}}{M-A_a}\right). \tag{9.14}$$

The result now follows by noting that, no matter how small T_a/T_1, $X_{s_3} \leqslant 1$ or, equivalently, $t_{s_3} \leqslant 2A_a^{-\frac{1}{2}}$.

(ii) *Perfectly rigid interface*

In the simple wave which is reflected from a perfectly rigid interface

$$A = \frac{X-X_c}{t-t_c}, \tag{9.15}$$

where

$$t_c = 2\left(\frac{1-A_a^{\frac{1}{2}}}{M-A_a}\right) \quad \text{and} \quad X_c = -1+Mt_c. \tag{9.16}$$

Consequently, *the reflected simple wave is also a centred wave*. This fact follows immediately if equations (7.7) and (7.11) are used to determine X and t as functions of A along the boundary curve $\bar{\bar{\beta}} = A_a^{-\frac{1}{2}}$ which separates the interaction region from the simple wave region. Note that for very weak waves, as $A_a \to 1$ the point at which this wave is centred $(X_c, t_c) \to (-1, 0)$. The variation of u in this centred wave follows from equations (9.10) and (9.15): these imply that

$$u = 2c_a - \hat{c}\left(\frac{X-X_c}{t-t_c}\right). \tag{9.17}$$

In equations (9.15) and (9.17)

$$\hat{A}(2c_a) \leqslant A \leqslant A_a \quad \text{and} \quad 0 \leqslant u/c_a \leqslant 1 \quad \text{when} \quad A_a \geqslant A_{\text{crit}},$$

while

$$0 \leqslant A \leqslant A_a \quad \text{and} \quad (2-c_1/c_a) \leqslant u/c_a \leqslant 1 \quad \text{when} \quad A_a \leqslant A_{\text{crit}}. \tag{9.18}$$

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